

VARIABLE-FREQUENCY NETWORK PERFORMANCE

LEARNING GOALS

Variable-Frequency Response Analysis

Network performance as function of frequency.
Transfer function

Sinusoidal Frequency Analysis

Bode plots to display frequency response data

Resonant Circuits

The resonance phenomenon and its characterization

Scaling

Impedance and frequency scaling

Filter Networks

Networks with frequency selective characteristics:
low-pass, high-pass, band-pass

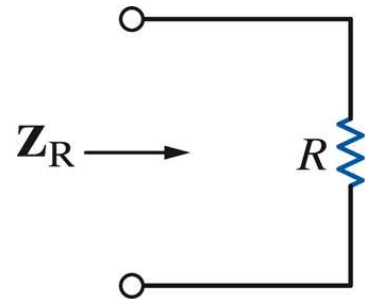


VARIABLE FREQUENCY-RESPONSE ANALYSIS

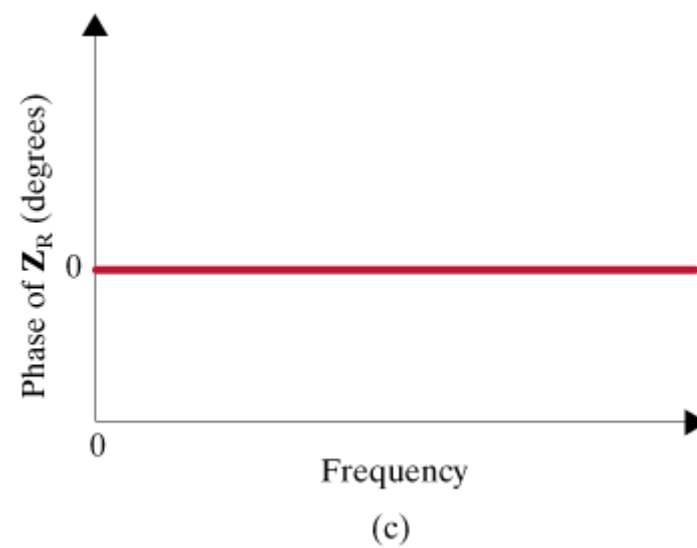
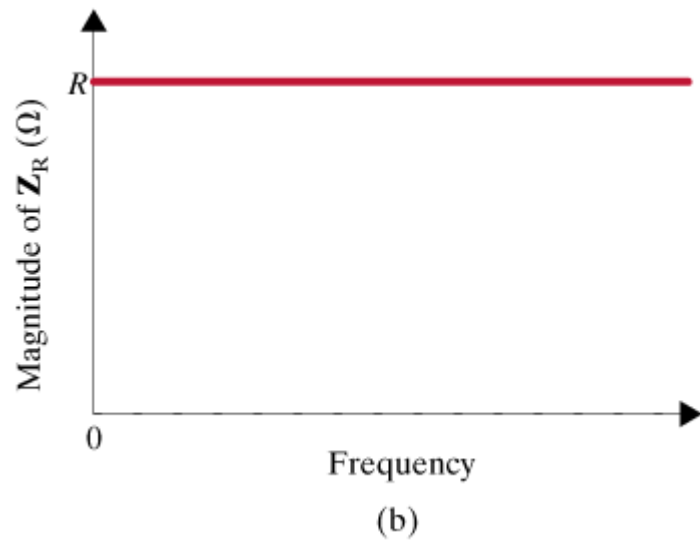
In AC steady state analysis the frequency is assumed constant (e.g., 60Hz). Here we consider the frequency as a variable and examine how the performance varies with the frequency.

Variation in impedance of basic components

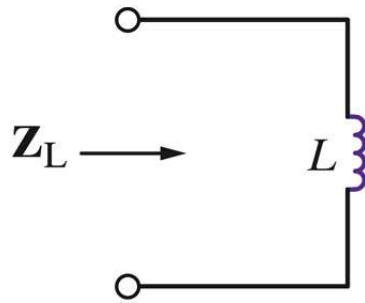
Resistor



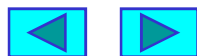
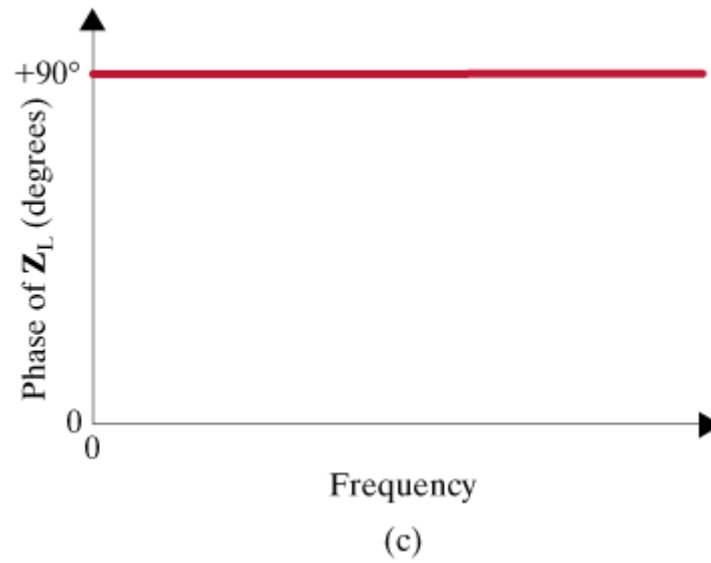
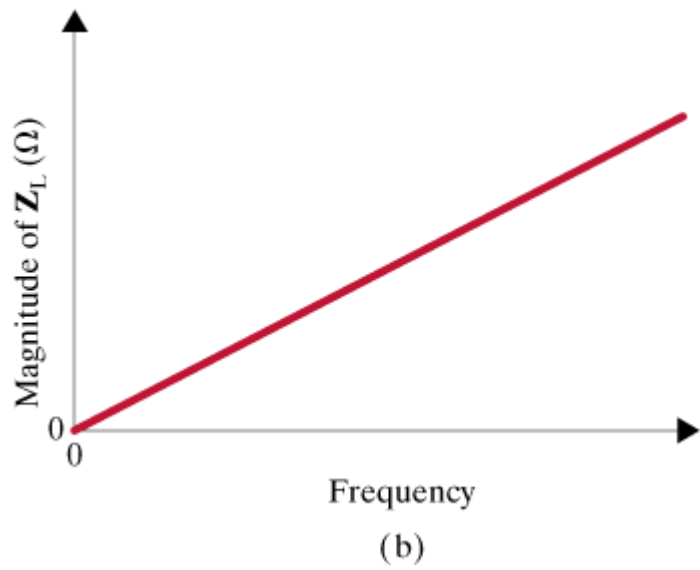
$$Z_R = R = R \angle 0^\circ$$



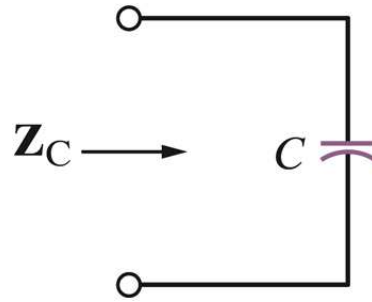
Inductor



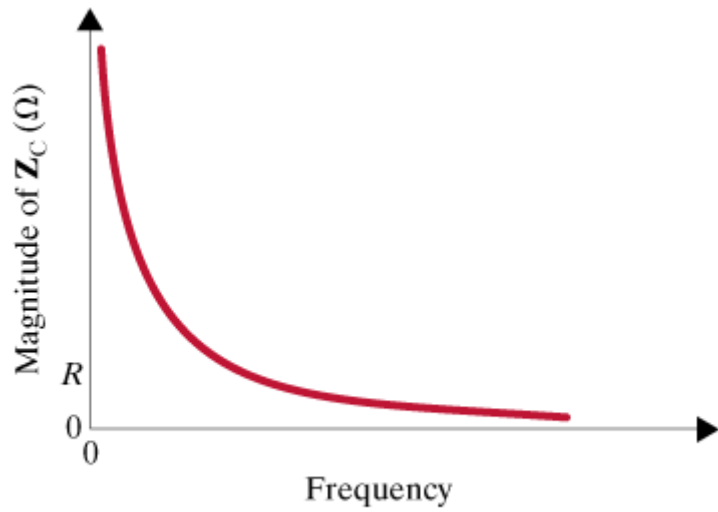
$$Z_L = j\omega L = \omega L \angle 90^\circ$$



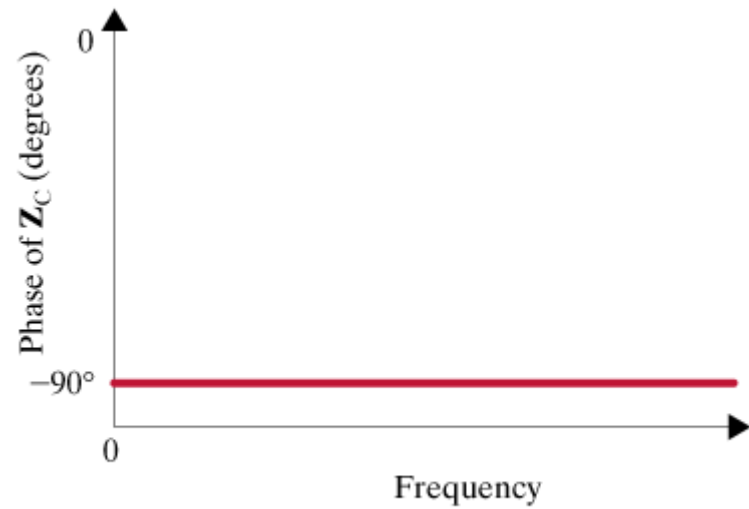
Capacitor



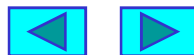
$$Z_c = \frac{1}{j\omega C} = \frac{1}{\omega C} \angle -90^\circ$$



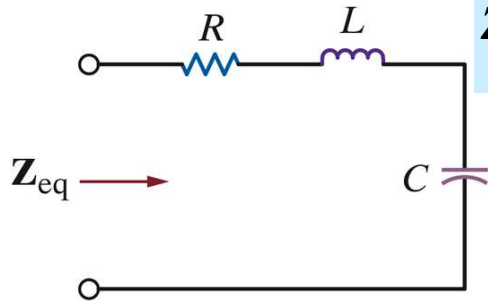
(b)



(c)



Frequency dependent behavior of series RLC network



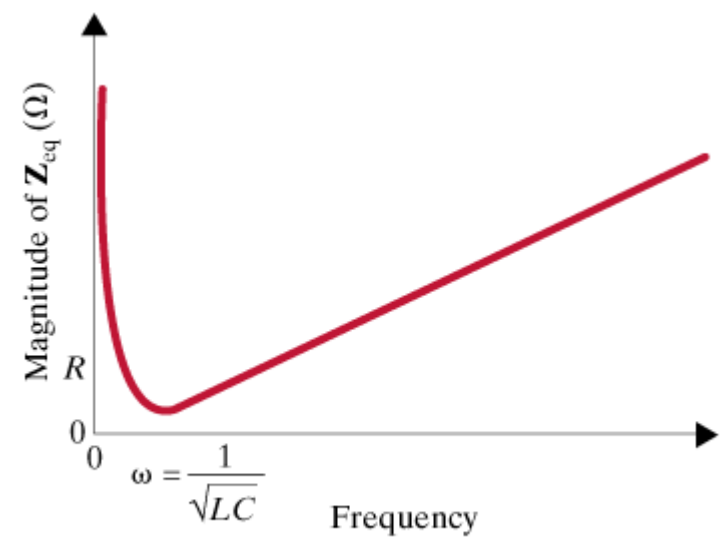
$$Z_{eq} = R + j\omega L + \frac{1}{j\omega C} = \frac{(j\omega)^2 LC + j\omega RC + 1}{j\omega C} \times \frac{-j}{-j} = \frac{\omega RC + j(\omega^2 LC - 1)}{\omega C}$$

"Simplification in notation" $j\omega \approx s$

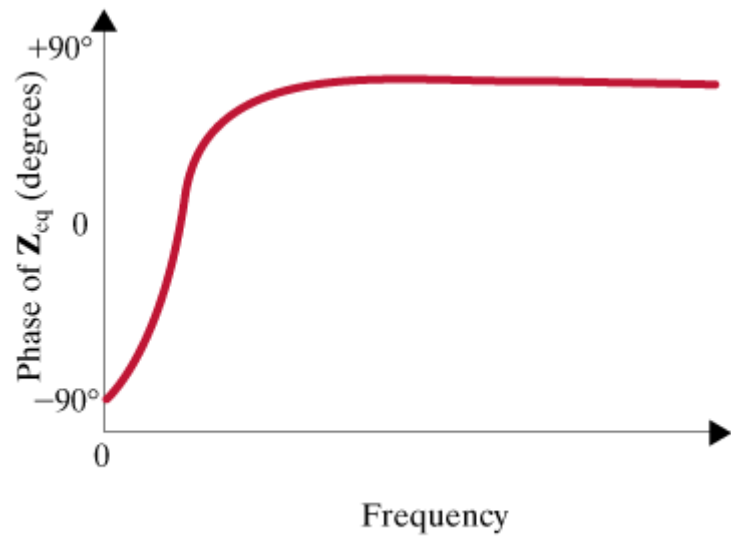
$$Z_{eq}(s) = \frac{s^2 LC + sRC + 1}{sC}$$

$$|Z_{eq}| = \frac{\sqrt{(\omega RC)^2 + (1 - \omega^2 LC)^2}}{\omega C}$$

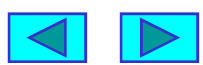
$$\angle Z_{eq} = \tan^{-1} \left(\frac{\omega^2 LC - 1}{\omega RC} \right)$$



(b)



(c)



Simplified notation for basic components

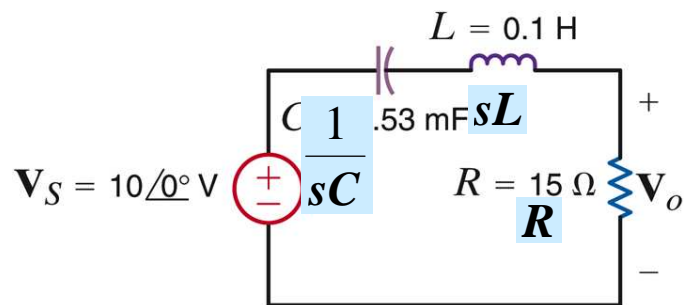
$$Z_R(s) = R, \quad Z_L(s) = sL, \quad Z_C = \frac{1}{sC}$$

For all cases seen, and all cases to be studied, the impedance is of the form

$$Z(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \dots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}$$

Moreover, if the circuit elements (L,R,C, dependent sources) are real then the expression for any voltage or current will also be a rational function in s

LEARNING EXAMPLE



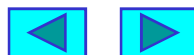
$$V_o(s) = \frac{R}{R + sL + 1/sC} V_S = \frac{sRC}{s^2 LC + sRC + 1} V_S$$

$$s \approx j\omega$$

$$V_o = \frac{j\omega RC}{(j\omega)^2 LC + j\omega RC + 1} V_S$$

$$V_o = \frac{j\omega(15 \times 2.53 \times 10^{-3})}{(j\omega)^2(0.1 \times 2.53 \times 10^{-3}) + j\omega(15 \times 2.53 \times 10^{-3}) + 1} 10 \angle 0^\circ$$

MATLAB can be effectively used to compute frequency response characteristics



USING MATLAB TO COMPUTE MAGNITUDE AND PHASE INFORMATION

$$V_o(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \dots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}$$

```
>> num = [a_m, a_{m-1}, ..., a_1, a_0];  
>> den = [b_n, b_{n-1}, ..., b_1, b_0];  
>> freqs(num, den)
```

MATLAB commands required to display magnitude and phase as function of frequency

NOTE: Instead of comma (,) one can use space to separate numbers in the array

EXAMPLE

$$V_o = \frac{j\omega(15 \times 2.53 \times 10^{-3})}{(j\omega)^2(0.1 \times 2.53 \times 10^{-3}) + j\omega(15 \times 2.53 \times 10^{-3}) + 1}$$

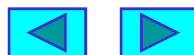
Diagram illustrating the mapping of coefficients from the transfer function to MATLAB arrays. Blue lines connect the coefficients in the numerator and denominator to their corresponding positions in the MATLAB arrays. The numerator coefficients are a_1 and a_0 . The denominator coefficients are b_2 , b_1 , and b_0 .

```
» num=[15*2.53*1e-3,0];  
» den=[0.1*2.53*1e-3,15*2.53*1e-3,1];  
» freqs(num,den)
```

Missing coefficients must be entered as zeros

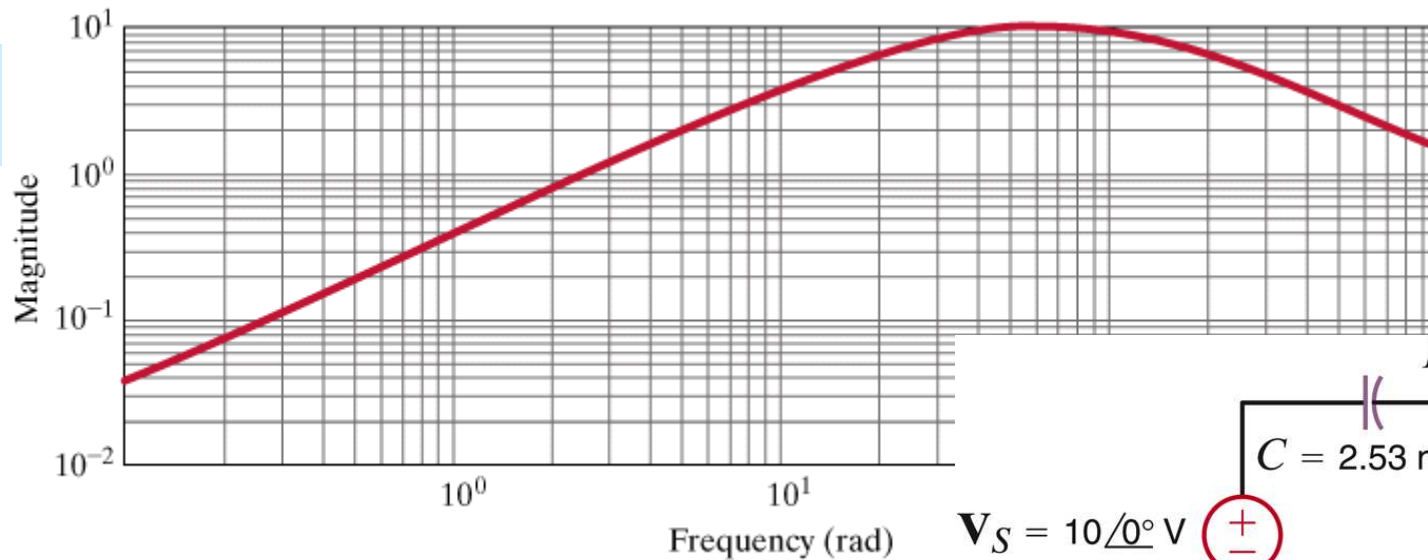
```
» num=[15*2.53*1e-3 0];  
» den=[0.1*2.53*1e-3 15*2.53*1e-3 1];  
» freqs(num,den)
```

This sequence will also work. Must be careful not to insert blanks elsewhere

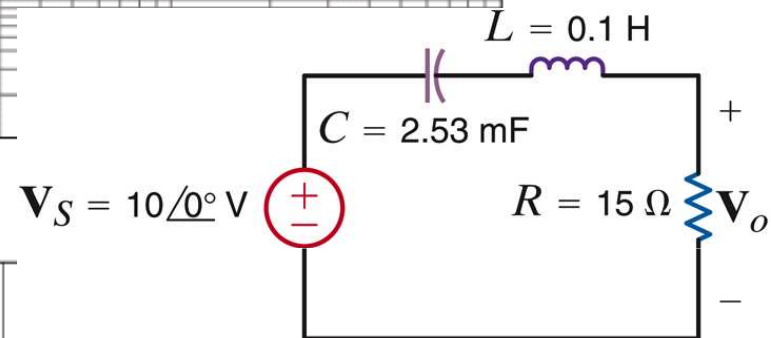
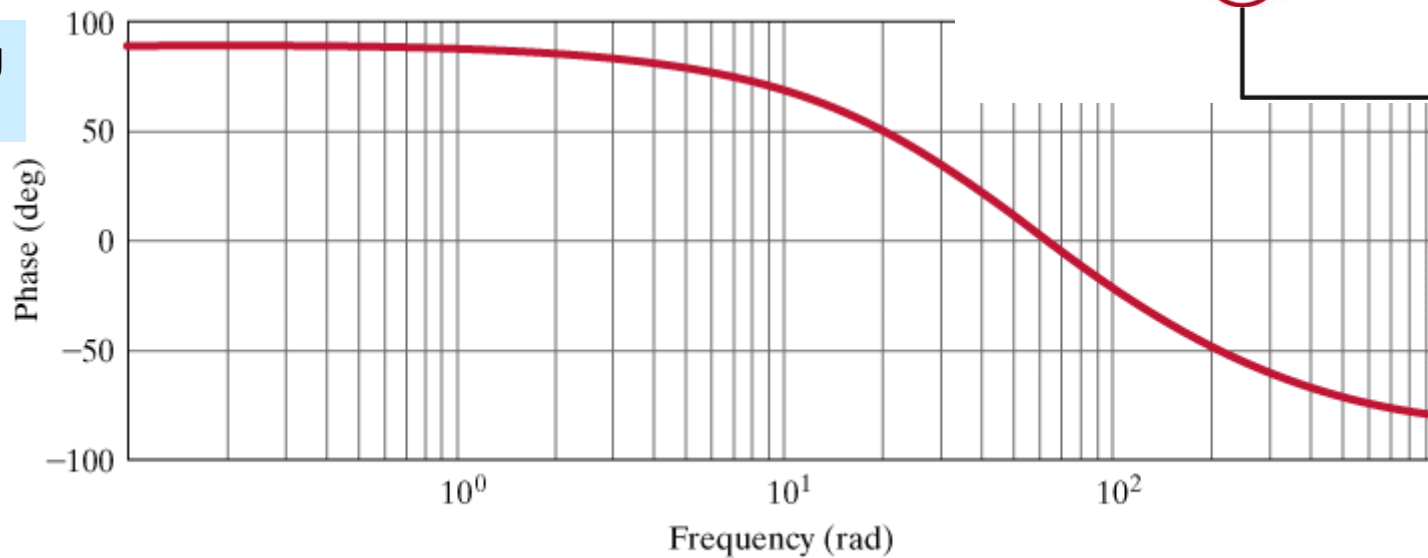


GRAPHIC OUTPUT PRODUCED BY MATLAB

Log-log plot



Semi-log plot

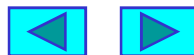
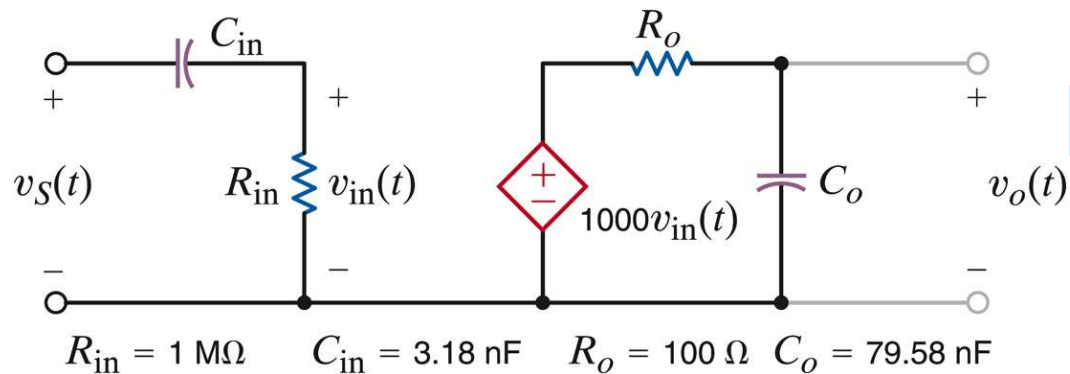
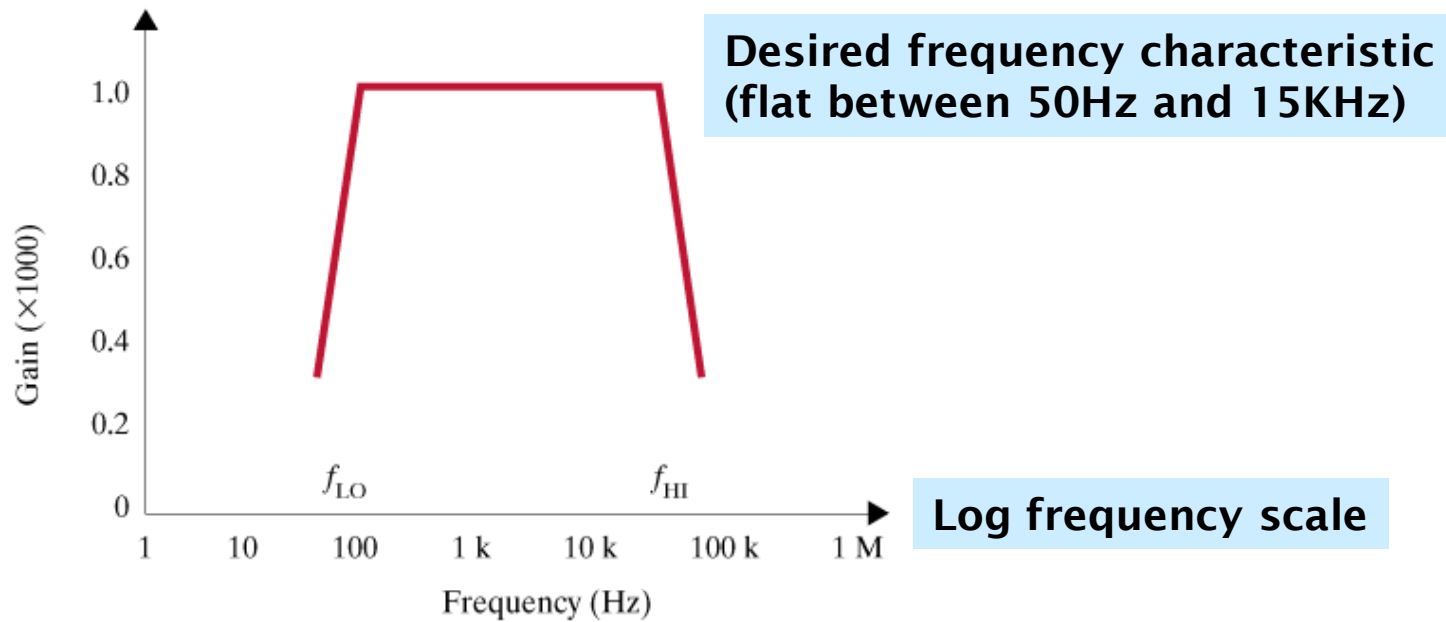


(b)



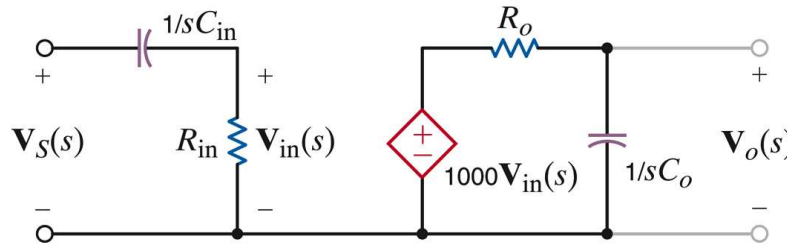
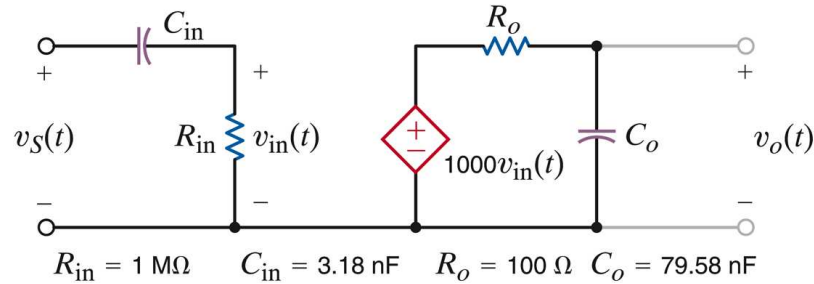
LEARNING EXAMPLE

A possible stereo amplifier



Frequency Analysis of Amplifier

$$V_{in}(s) = \frac{R_{in}}{R_{in} + 1/sC_{in}} V_S(s)$$



$$G(s) = \frac{V_o(s)}{V_S(s)} = \frac{V_{in}(s)}{V_S(s)} \frac{V_o(s)}{V_{in}(s)}$$

Voltage Gain

$$V_o(s) = \frac{1/sC_o}{1/sC_o + R_o} [1000V_{in}]$$

Frequency domain equivalent circuit

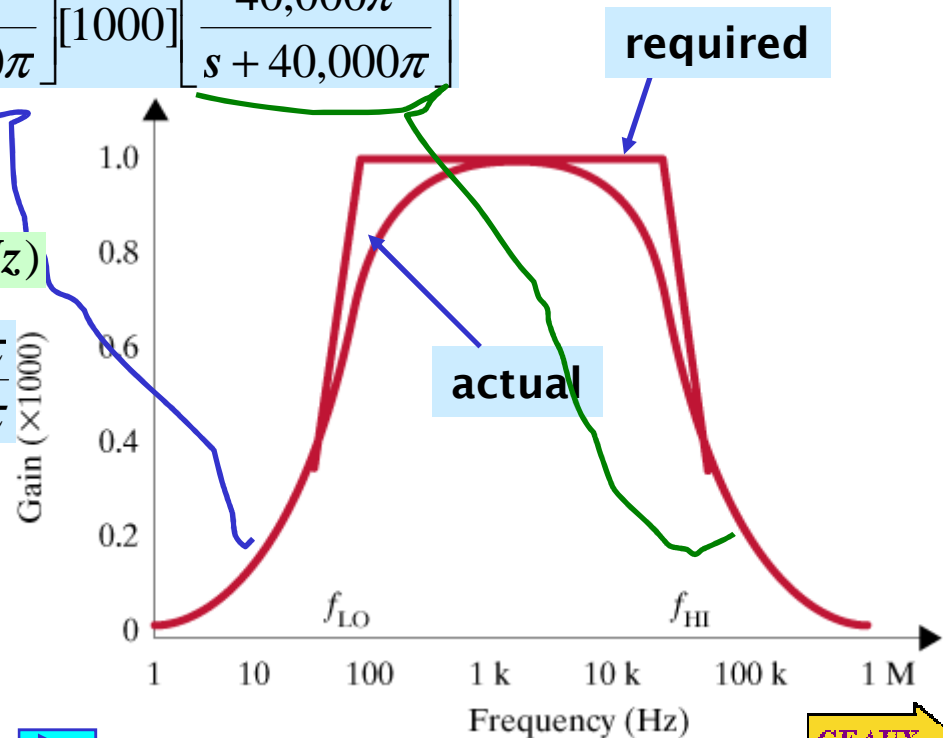
$$G(s) = \left[\frac{sC_{in}R_{in}}{1 + sC_{in}R_{in}} \right] [1000] \left[\frac{1}{1 + sC_oR_o} \right] = \left[\frac{s}{s + 100\pi} \right] [1000] \left[\frac{40,000\pi}{s + 40,000\pi} \right]$$

$$(C_{in}R_{in})^{-1} = (3.18 \times 10^{-9} \times 10^6)^{-1} \approx 100\pi \text{ (50Hz)}$$

$$(C_oR_o)^{-1} = (79.58 \times 10^{-9} \times 100)^{-1} \approx 40,000\pi \text{ (20kHz)}$$

$$100\pi \ll |s| \ll 40,000\pi \Rightarrow G(s) \approx \frac{s}{s} [1000] \frac{40,000\pi}{40,000\pi} \times 1000$$

Frequency dependent behavior is caused by reactive elements



NETWORK FUNCTIONS

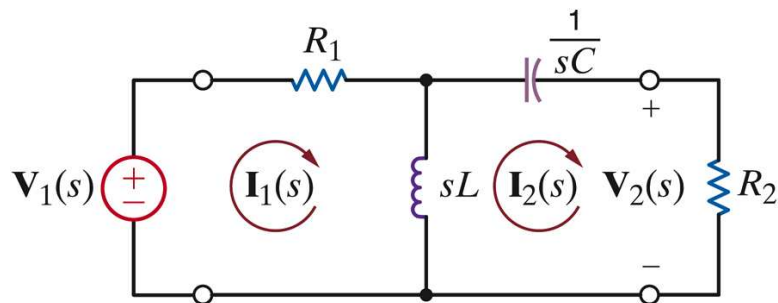
Some nomenclature

When voltages and currents are defined at different terminal pairs we define the ratios as **Transfer Functions**

INPUT	OUTPUT	TRANSFER FUNCTION	SYMBOL
Voltage	Voltage	Voltage Gain	$G_v(s)$
Current	Voltage	Transimpedance	$Z(s)$
Current	Current	Current Gain	$G_i(s)$
Voltage	Current	Transadmittance	$Y(s)$

If voltage and current are defined at the same terminals we define **Driving Point Impedance/Admittance**

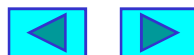
EXAMPLE



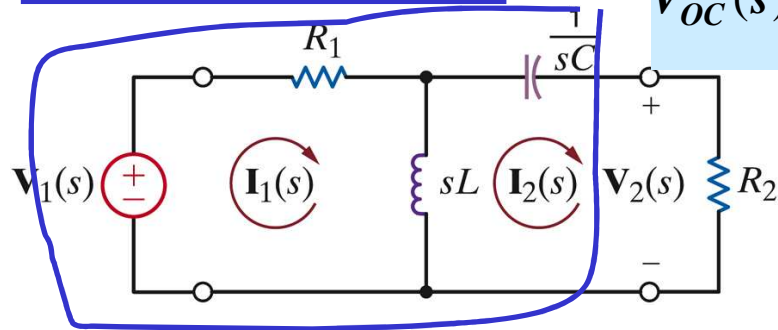
To compute the transfer functions one must solve the circuit. Any valid technique is acceptable

$$Y_T(s) = \frac{I_2(s)}{V_1(s)} \begin{cases} \text{Transadmittance} \\ \text{Transfer admittance} \end{cases}$$

$$G_v(s) = \frac{V_2(s)}{V_1(s)} \quad \text{Voltage gain}$$



LEARNING EXAMPLE



$$V_{OC}(s) = \frac{sL}{sL + R_1} V_1(s)$$

The textbook uses mesh analysis. We will use Thevenin's theorem

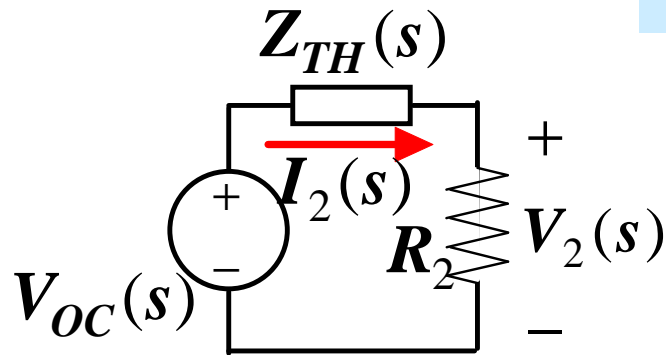
$$Z_{TH}(s) = \frac{1}{sC} + R_1 \parallel sL = \frac{1}{sC} + \frac{sLR_1}{sL + R_1}$$

$$Z_{TH}(s) = \frac{s^2 LCR_1 + sL + R_1}{sC(sL + R_1)}$$

$$Y_T(s) = \frac{I_2(s)}{V_1(s)} \begin{cases} \text{Transadmittance} \\ \text{Transfer admittance} \end{cases}$$

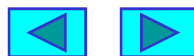
$$G_v(s) = \frac{V_2(s)}{V_1(s)} \quad \text{Voltage gain}$$

$$I_2(s) = \frac{V_{OC}(s)}{R_2 + Z_{TH}(s)} = \frac{\frac{sL}{sL + R_1} V_1(s)}{R_2 + \frac{s^2 LCR_1 + sL + R_1}{sC(sL + R_1)}} \times \frac{sC(sL + R_1)}{sC(sL + R_1)}$$



$$Y_T(s) = \frac{s^2 LC}{s^2 (R_1 + R_2) LC + s(L + R_1 R_2 C) + R_1}$$

$$G_v(s) = \frac{V_s(s)}{V_1(s)} = \frac{R_2 I_2(s)}{V_1(s)} = R_2 Y_T(s)$$



POLES AND ZEROS

(More nomenclature)

$$H(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \dots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}$$

Arbitrary network function

Using the roots, every (monic) polynomial can be expressed as a product of first order terms

$$H(s) = K_0 \frac{(s - z_1)(s - z_2)\dots(s - z_m)}{(s - p_1)(s - p_2)\dots(s - p_n)}$$

z_1, z_2, \dots, z_m = zeros of the network function

p_1, p_2, \dots, p_n = poles of the network function

The network function is uniquely determined by its poles and zeros and its value at some other value of s (to compute the gain)

EXAMPLE

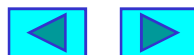
zeros : $z_1 = -1$,

poles : $p_1 = -2 + j2, p_2 = -2 - j2$

$H(0) = 1$

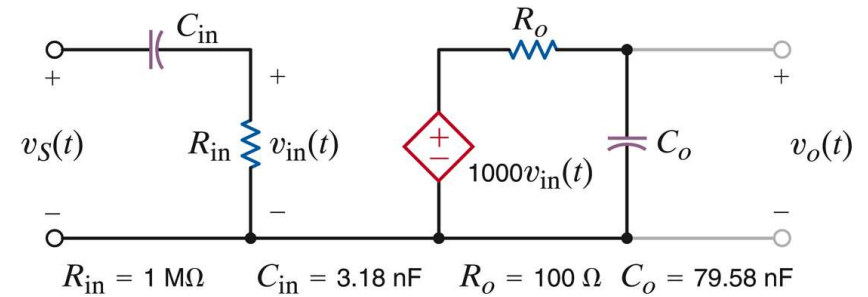
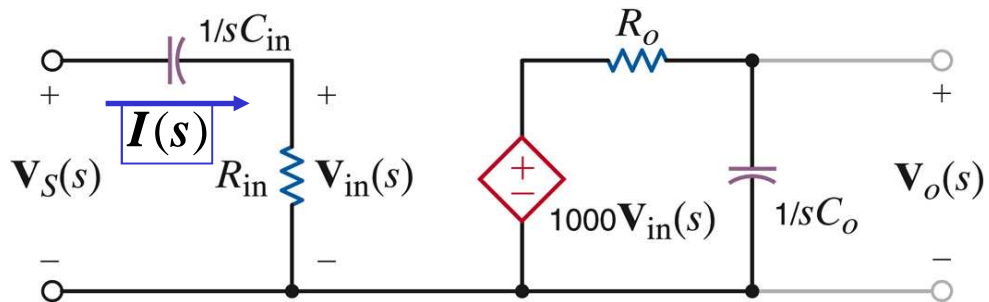
$$H(s) = K_0 \frac{(s + 1)}{(s + 2 - j2)(s + 2 + j2)} = K_0 \frac{s + 1}{s^2 + 4s + 8}$$

$$H(0) = K_0 \frac{1}{8} = 1 \Rightarrow H(s) = 8 \frac{s + 1}{s^2 + 4s + 8}$$



LEARNING EXTENSION

Find the driving point impedance at $V_S(s)$

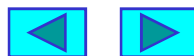


$$Z(s) = \frac{V_S(s)}{I(s)}$$

$$\text{KVL : } V_S(s) = R_{in} I(s) + \frac{1}{sC_{in}} I(s)$$

$$Z(s) = R_{in} + \frac{1}{sC_{in}} = \left[1 + \frac{100\pi}{s} \right] \text{ M}\Omega$$

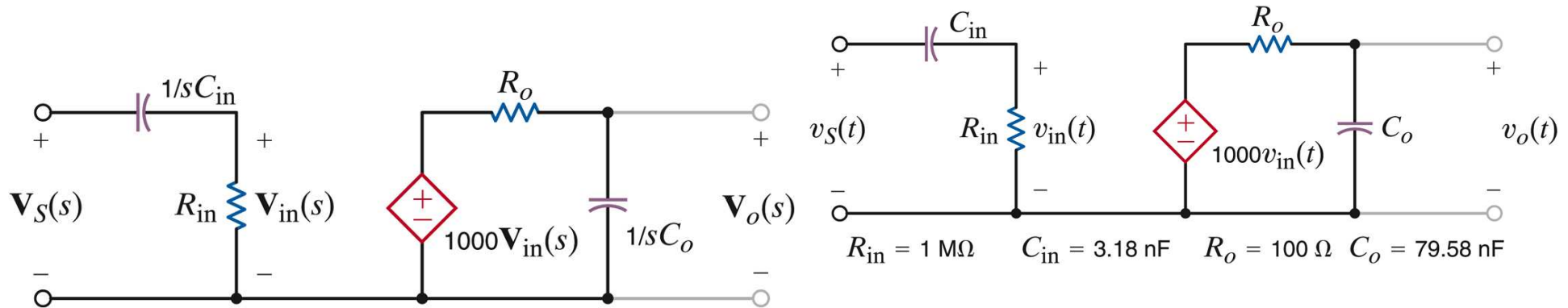
Replace numerical values



LEARNING EXTENSION

Find the pole and zero locations and the value of K_0

for the voltage gain $G(s) = \frac{V_o(s)}{V_S(s)}$



$$H(s) = K_0 \frac{(s - z_1)(s - z_2)\dots(s - z_m)}{(s - p_1)(s - p_2)\dots(s - p_n)}$$

Zeros = roots of numerator
Poles = roots of denominator

For this case the gain was shown to be

$$G(s) = \left[\frac{sC_{in}R_{in}}{1 + sC_{in}R_{in}} \right] [1000] \left[\frac{1}{1 + sC_oR_o} \right] = \left[\frac{s}{s + 100\pi} \right] [1000] \left[\frac{40,000\pi}{s + 40,000\pi} \right]$$

zero: $z_1 = 0$

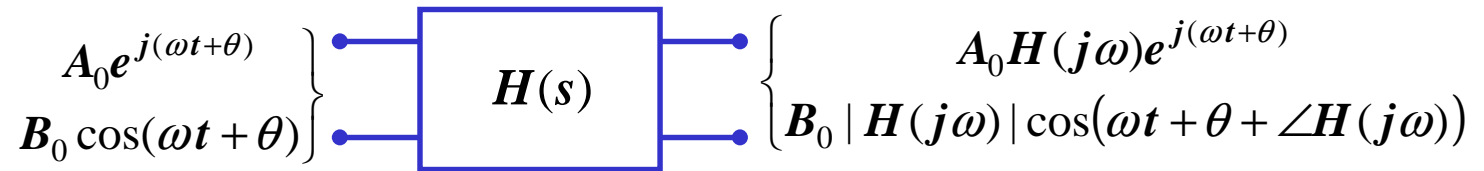
poles: $p_1 = -50\text{Hz}$, $p_2 = -20,000\text{Hz}$

$K_0 = (4 \times 10^7)\pi$

Variable
Frequency
Response



SINUSOIDAL FREQUENCY ANALYSIS



Circuit represented by
network function

To study the behavior of a network as a function of the frequency we analyze the network function $H(j\omega)$ as a function of ω .

Notation

$$M(\omega) = |H(j\omega)|$$

$$\phi(\omega) = \angle H(j\omega)$$

$$H(j\omega) = M(\omega) e^{j\phi(\omega)}$$

Plots of $M(\omega)$, $\phi(\omega)$, as function of ω are generally called magnitude and phase characteristics.

BODE PLOTS $\left\{ \begin{array}{l} 20 \log_{10}(M(\omega)) \\ \phi(\omega) \end{array} \right.$ vs $\log_{10}(\omega)$



HISTORY OF THE DECIBEL

Originated as a measure of relative (radio) power

$$P_2 |_{dB} \text{ (over } P_1) = 10 \log \frac{P_2}{P_1}$$

$$P = I^2 R = \frac{V^2}{R} \Rightarrow P_2 |_{dB} \text{ (over } P_1) = 10 \log \frac{V_2^2}{V_1^2} = 10 \log \frac{I_2^2}{I_1^2}$$

By extension

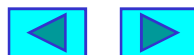
$$V |_{dB} = 20 \log_{10} |V|$$

$$I |_{dB} = 20 \log_{10} |I|$$

$$G |_{dB} = 20 \log_{10} |G|$$

Using log scales the frequency characteristics of network functions have simple asymptotic behavior.

The asymptotes can be used as reasonable and efficient approximations



General form of a network function showing basic terms

Frequency independent

Poles/zeros at the origin

$$H(j\omega) = \frac{K_0 (j\omega)^{\pm N} (1 + j\omega\tau_1) [1 + 2\zeta_3(j\omega\tau_3) + (j\omega\tau_3)^2] \dots}{(1 + j\omega\tau_a) [1 + 2\zeta_b(j\omega\tau_b) + (j\omega\tau_b)^2] \dots}$$

$$\log(AB) = \log A + \log B$$

$$\log\left(\frac{N}{D}\right) = \log N - \log D$$

First order terms

Quadratic terms for complex conjugate poles/zeros

$$|H(j\omega)|_{dB} = 20\log_{10} |H(j\omega)| = 20\log_{10} K_0 \pm N 20\log_{10} |j\omega|$$

$$+ 20\log_{10} |1 + j\omega\tau_1| + 20\log_{10} |1 + 2\zeta_3(j\omega\tau_3) + (j\omega\tau_3)^2| + \dots$$

$$- 20\log_{10} |1 + j\omega\tau_a| - 20\log_{10} |1 + 2\zeta_b(j\omega\tau_b) + (j\omega\tau_b)^2| - \dots$$

$$\angle z_1 z_2 = \angle z_1 + \angle z_2$$

$$\angle \frac{z_1}{z_2} = \angle z_1 - \angle z_2$$

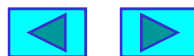
$$\angle H(j\omega) = 0 \pm N 90^\circ$$

$$+ \tan^{-1} \omega\tau_1 + \tan^{-1} \frac{2\zeta_3\omega\tau_3}{1 - (\omega\tau_3)^2} + \dots$$

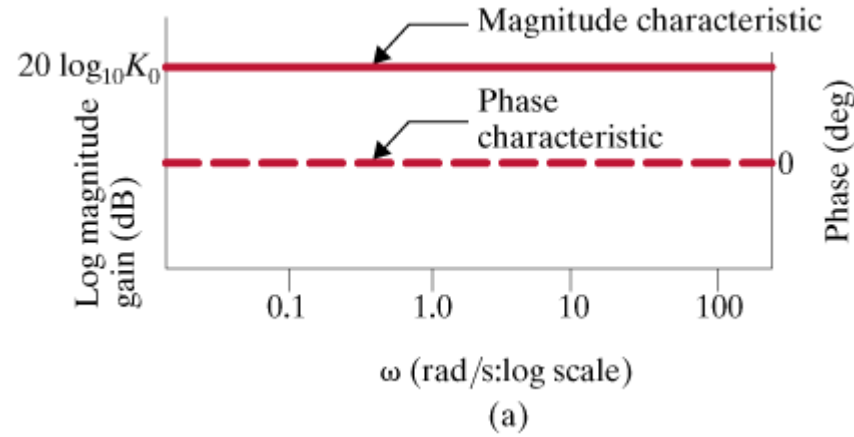
$$- \tan^{-1} \omega\tau_a - \tan^{-1} \frac{2\zeta_b\omega\tau_b}{1 - (\omega\tau_b)^2} - \dots$$

Display each basic term separately and add the results to obtain final answer

Let's examine each basic term



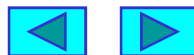
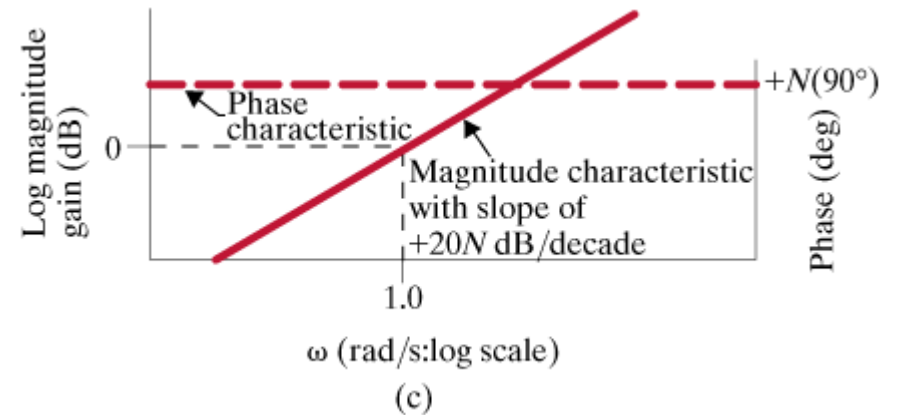
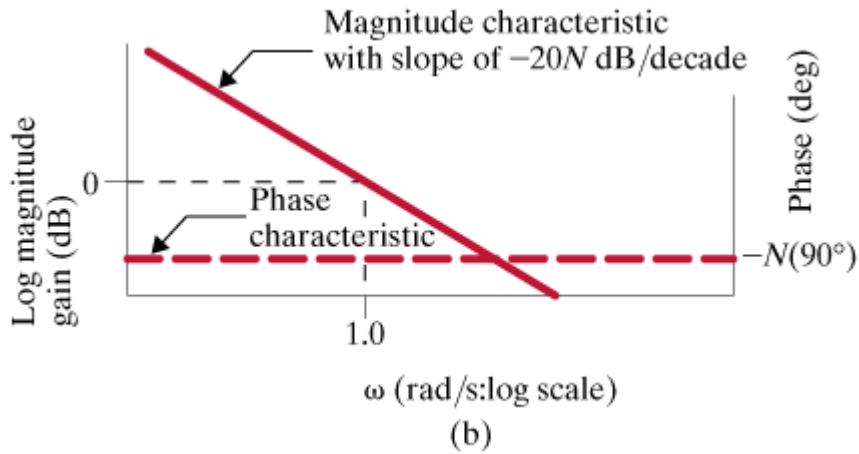
Constant Term



the x - axis is $\log_{10}\omega$
this is a straight line

Poles/Zeros at the origin

$$(j\omega)^{\pm N} \rightarrow \begin{cases} |(j\omega)^{\pm N}|_{dB} = \pm N \times 20 \log_{10}(\omega) \\ \angle(j\omega)^{\pm N} = \pm N 90^\circ \end{cases}$$



Simple pole or zero $1 + j\omega\tau$ $\left\{ \begin{array}{l} |1 + j\omega\tau|_{dB} = 20 \log_{10} \sqrt{1 + (\omega\tau)^2} \\ \angle(1 + j\omega\tau) = \tan^{-1} \omega\tau \end{array} \right.$

$\omega\tau \ll 1 \Rightarrow |1 + j\omega\tau|_{dB} \approx 0$ low frequency asymptote

$\angle(1 + j\omega\tau) \approx 0^\circ$

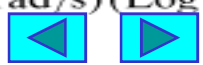
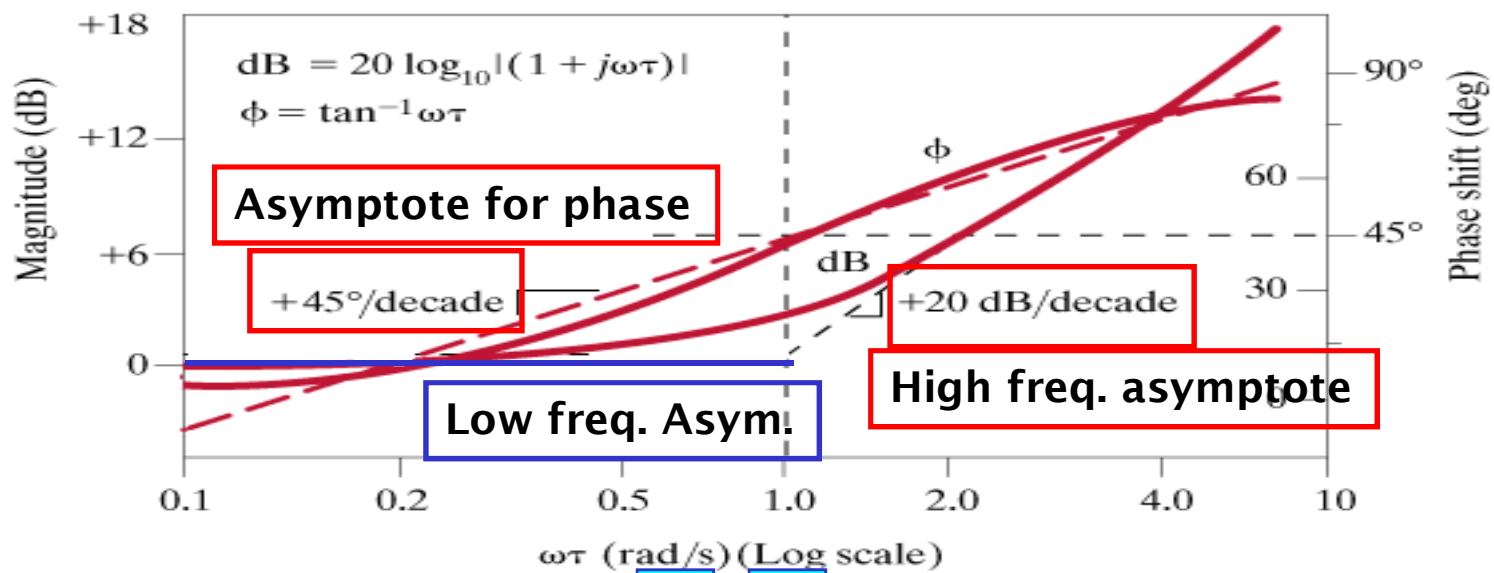
$\omega\tau \gg 1 \Rightarrow |1 + j\omega\tau|_{dB} \approx 20 \log_{10} \omega\tau$ high frequency asymptote (20dB/dec)

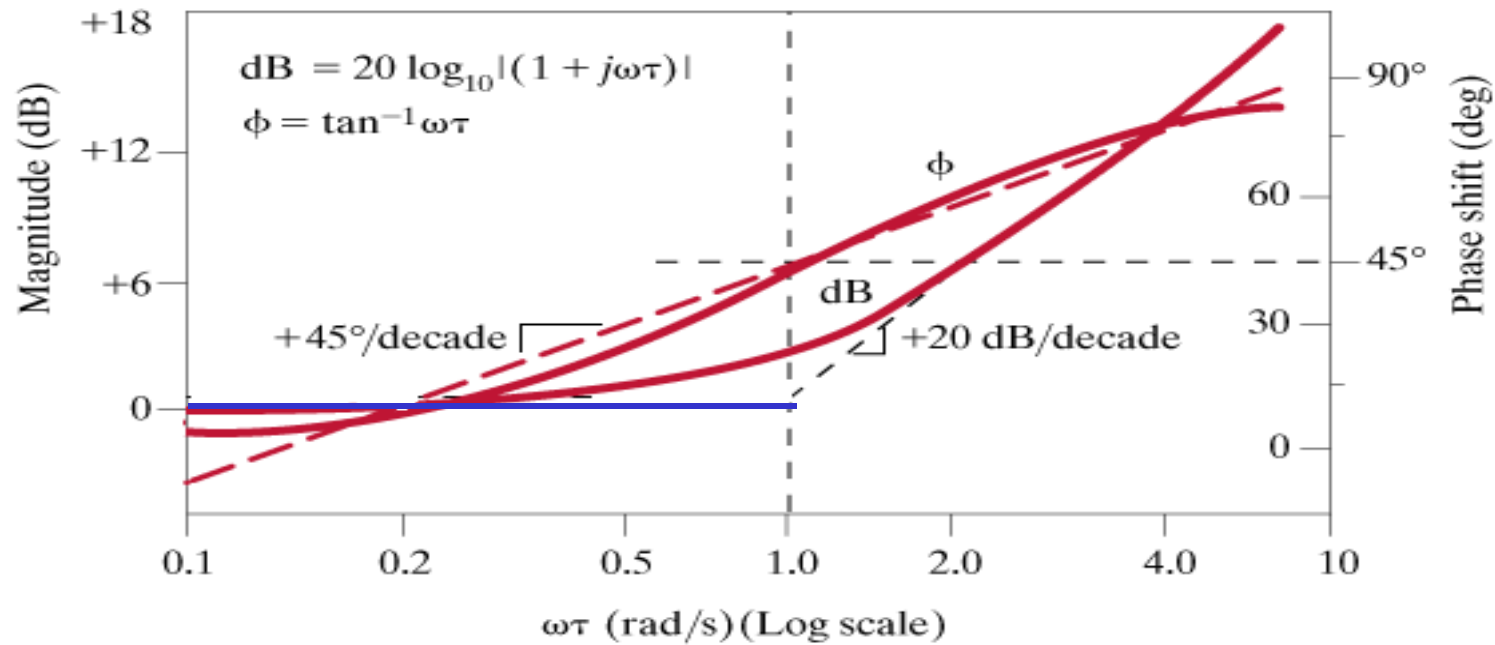
$\angle(1 + j\omega\tau) \approx 90^\circ$

The two asymptotes meet when $\omega\tau = 1$ (corner/break frequency)

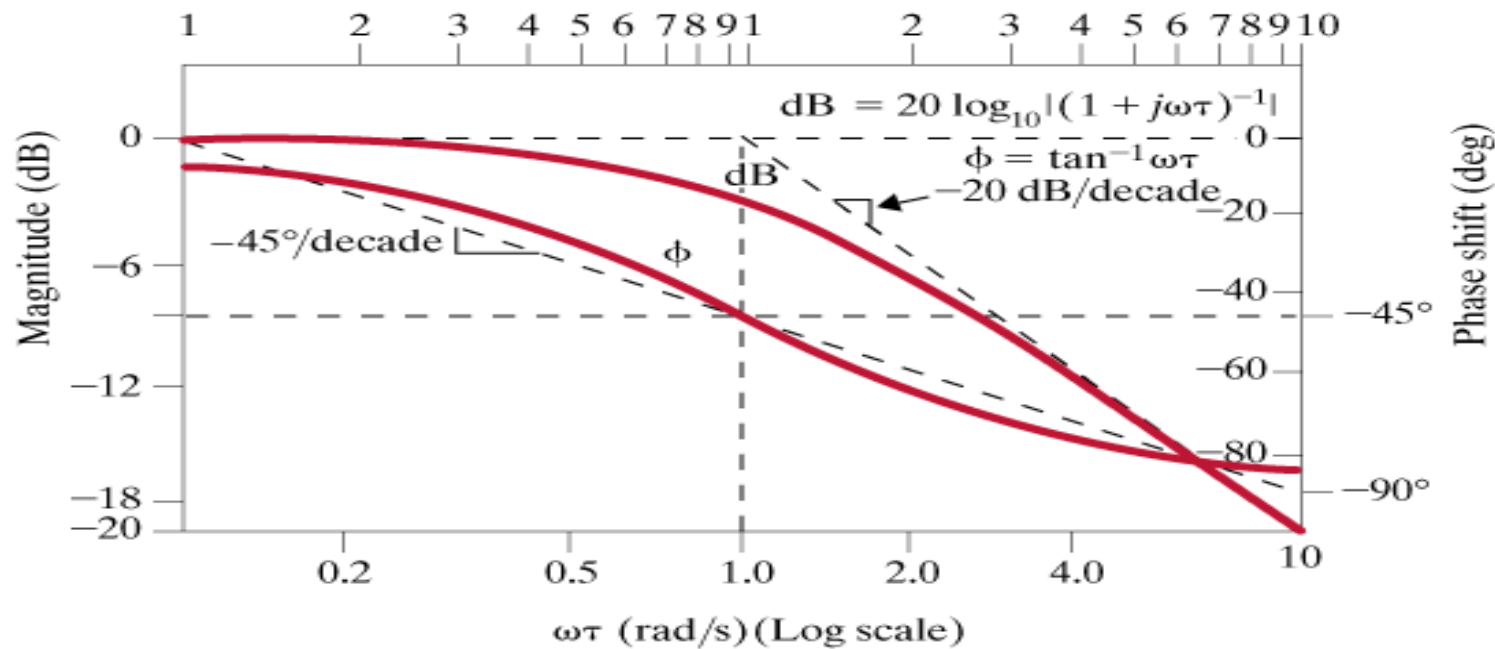
Behavior in the neighborhood of the corner

	Frequency	Asymptote	Curve	distance to asymptote	Argument
corner	$\omega\tau = 1$	0dB	3dB	3	45
octave above	$\omega\tau = 2$	6dB	7db	1	63.4
octave below	$\omega\tau = 0.5$	0dB	1dB	1	26.6





Simple zero



Simple pole

(a) ◀ ▶



Quadratic pole or zero $t_2 = [1 + 2\zeta(j\omega\tau) + (j\omega\tau)^2] = [1 + 2\zeta(j\omega\tau) - (\omega\tau)^2]$

$$|t_2|_{dB} = 20 \log_{10} \sqrt{(1 - (\omega\tau)^2)^2 + (2\zeta\omega\tau)^2}$$

$$\angle t_2 = \tan^{-1} \frac{2\zeta\omega\tau}{1 - (\omega\tau)^2}$$

$\omega\tau \ll 1$ $|t_2|_{dB} \approx 0$ low frequency asymptote

$$\angle t_2 \approx 0^\circ$$

$\omega\tau \gg 1$ $|t_2|_{dB} \approx 20 \log_{10} (\omega\tau)^2$ high freq. asymptote **40dB/dec**

$$\angle t_2 \approx 180^\circ$$

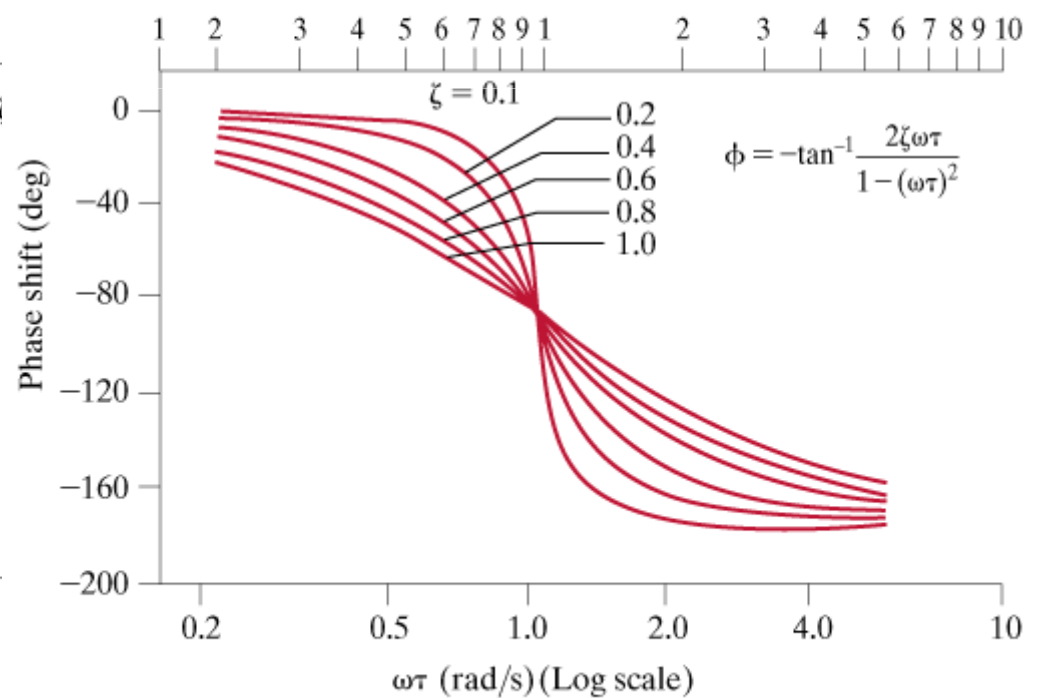
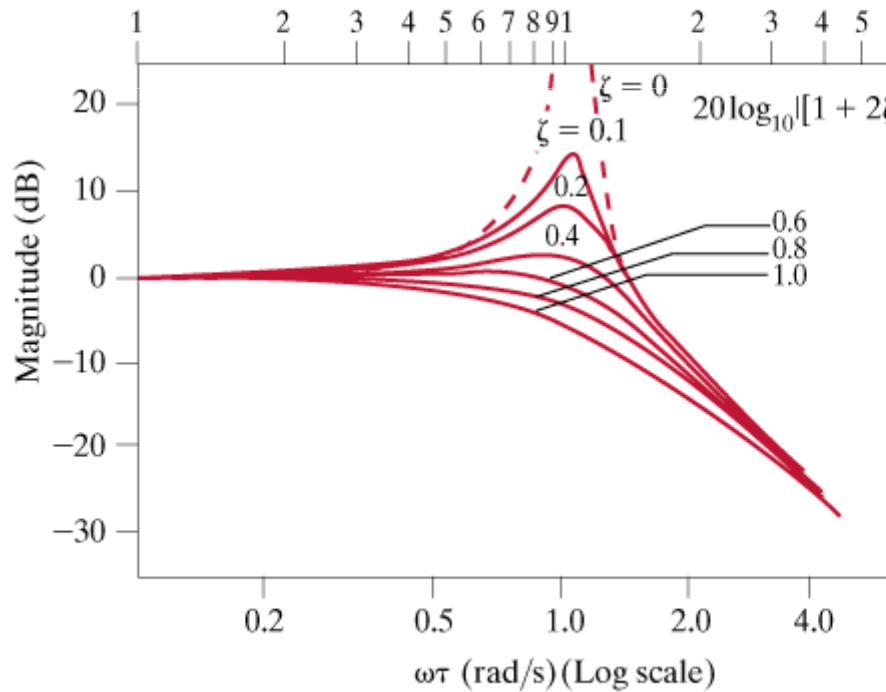
$\omega\tau = 1$ $|t_2|_{dB} = 20 \log_{10} (2\zeta)$ **Corner/break frequency**

$$\angle t_2 = 90^\circ$$

$\omega\tau = \sqrt{1 - 2\zeta^2}$ $|t_2|_{dB} = 20 \log_{10} 2\zeta\sqrt{1 - \zeta^2}$ **Resonance frequency**

$$\angle t_2 = \tan^{-1} \frac{\sqrt{1 - 2\zeta^2}}{\zeta} \quad \zeta \leq \frac{\sqrt{2}}{2}$$

These graphs are inverted for a zero



Magnitude for quadratic pole



Phase for quadratic pole



LEARNING EXAMPLE

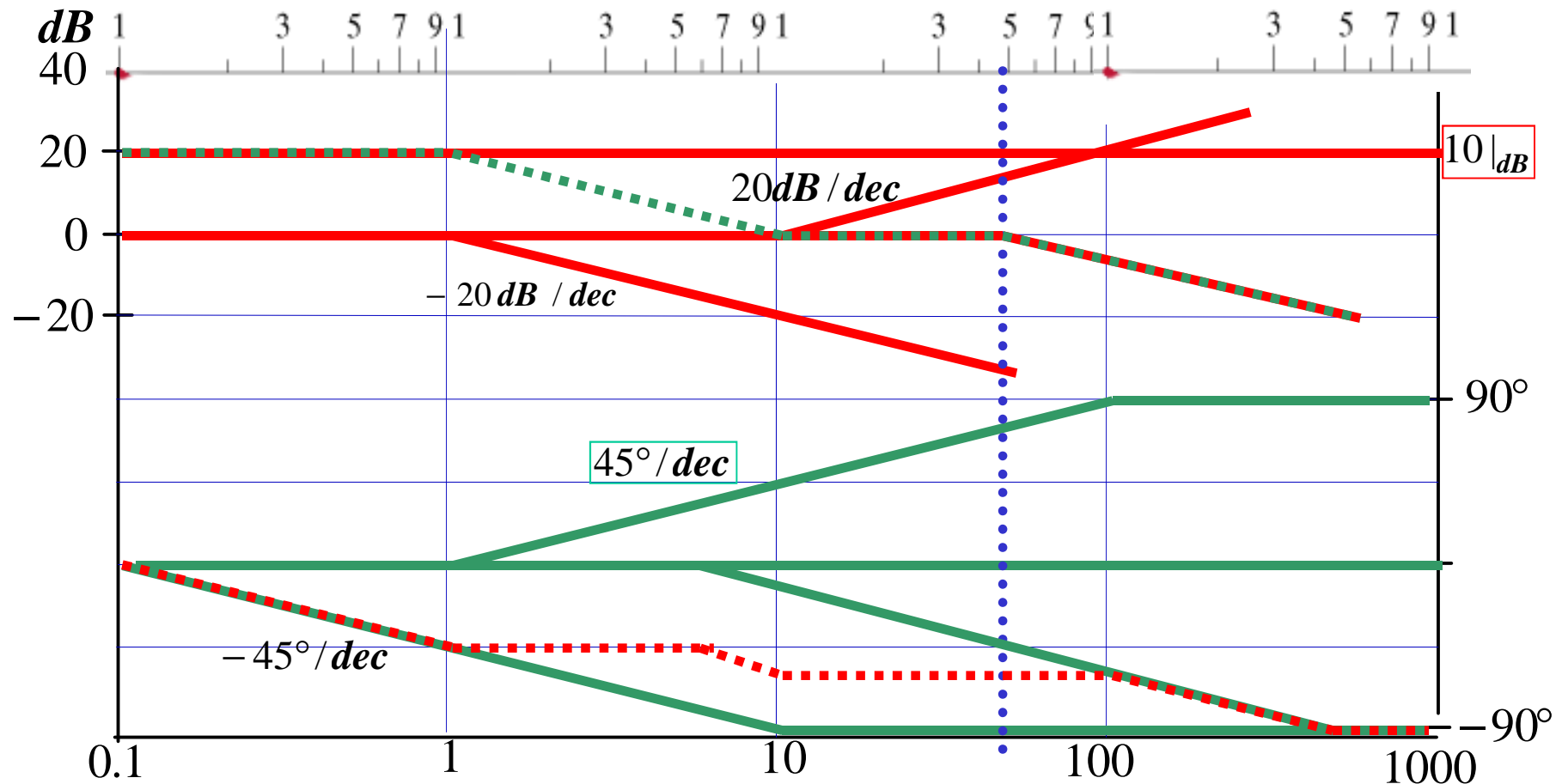
Generate magnitude and phase plots

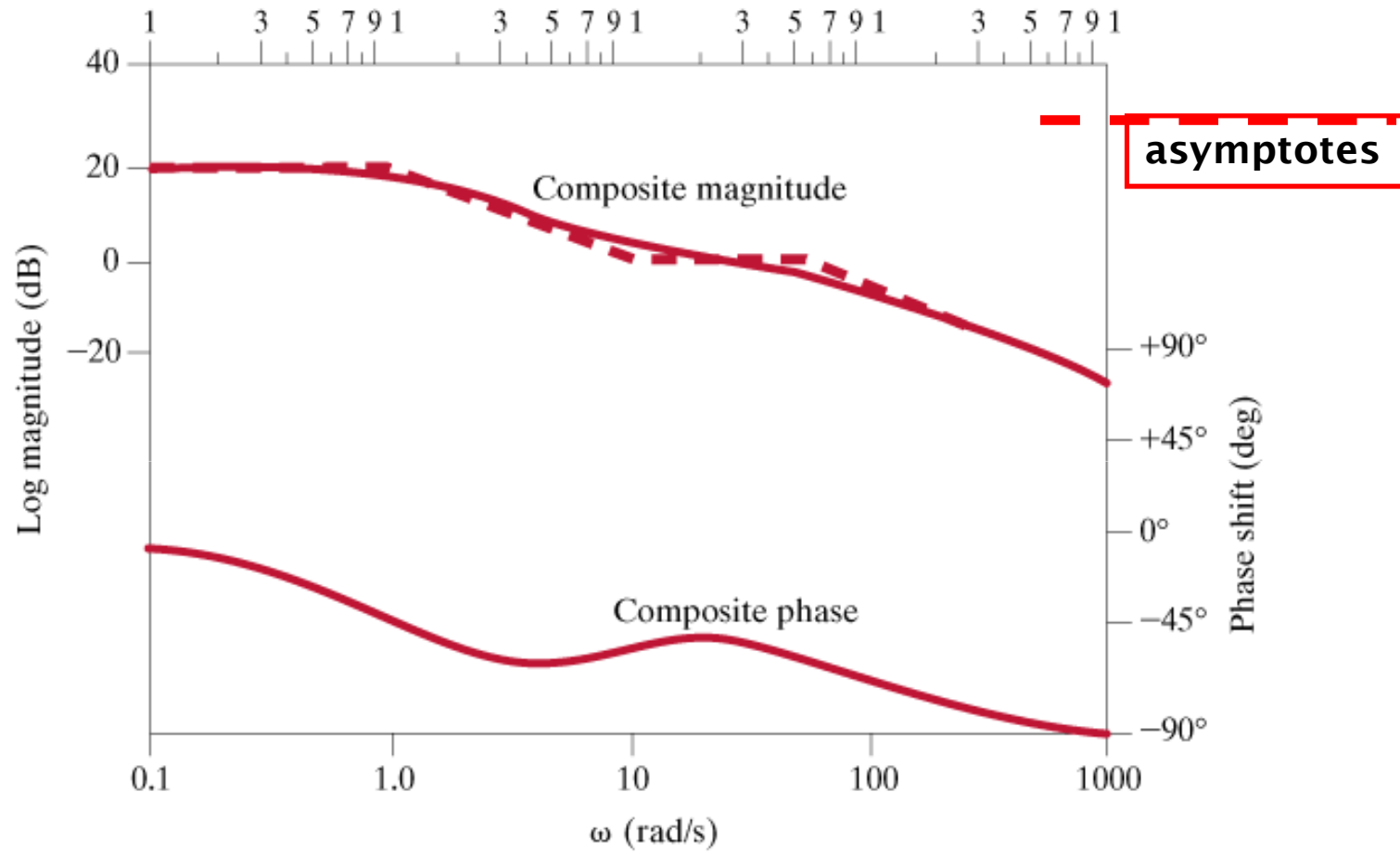
Draw asymptotes for each term

$$G_v(j\omega) = \frac{10(0.1j\omega + 1)}{(j\omega + 1)(0.02j\omega + 1)}$$

Breaks/corners: 1, 10, 50

Draw composites





(b)



LEARNING EXAMPLE

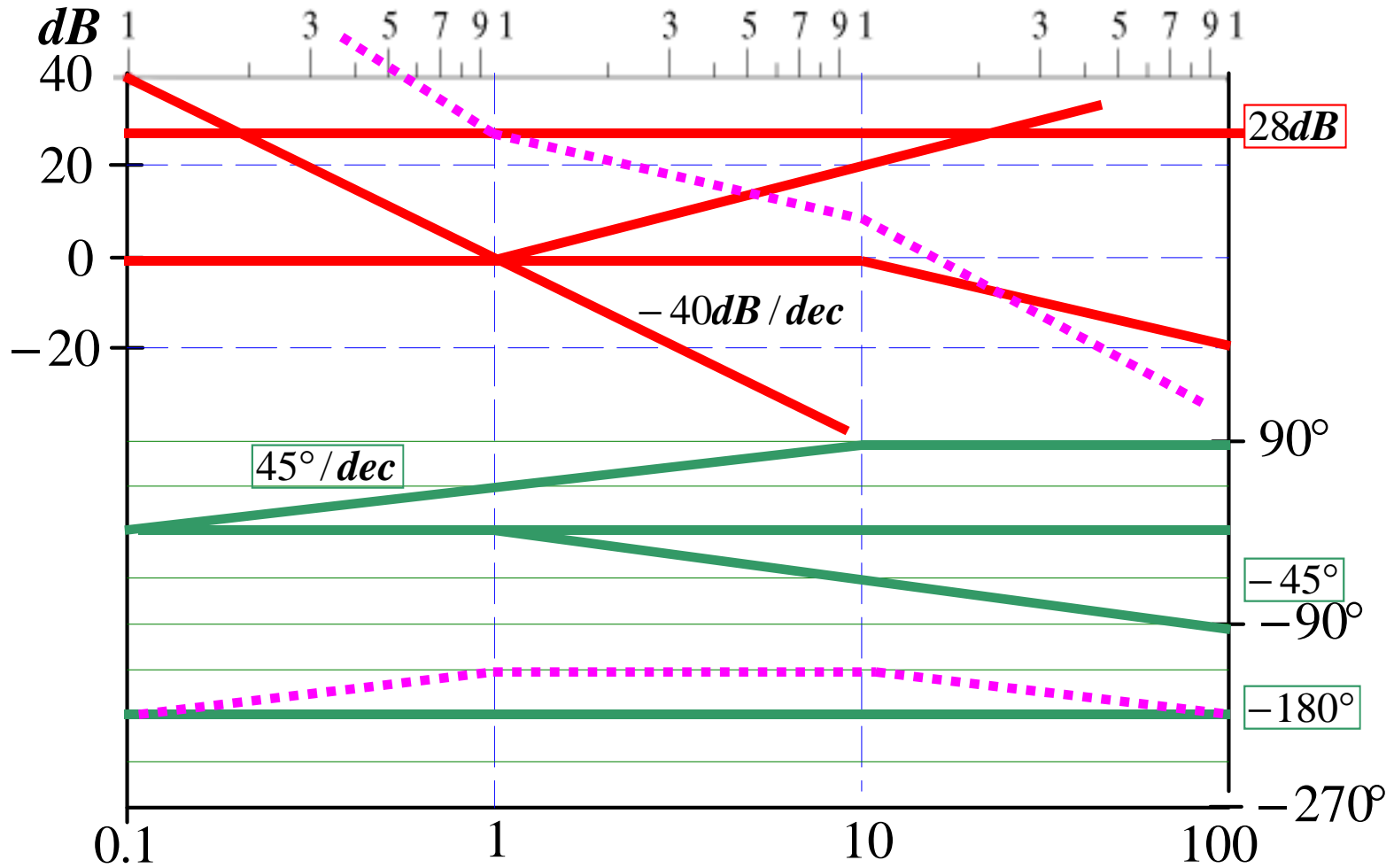
Generate magnitude and phase plots

Draw asymptotes for each

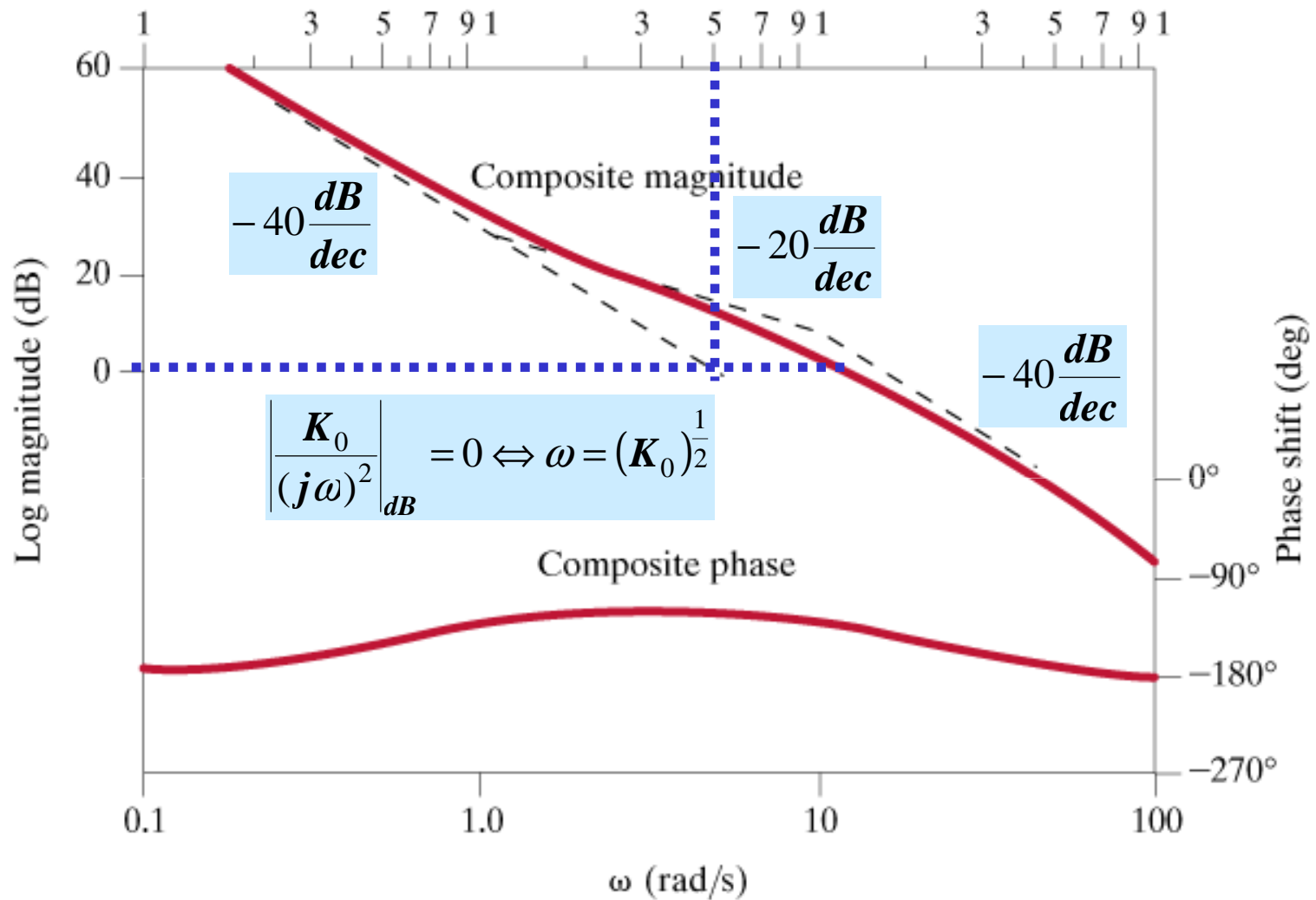
$$G_v(j\omega) = \frac{25(j\omega + 1)}{(j\omega)^2(0.1j\omega + 1)}$$

Breaks (corners): 1, 10

Form composites



Final results . . . And an extra hint on poles at the origin



(b)



LEARNING EXTENSION

Sketch the magnitude characteristic

$$G(j\omega) = \frac{10^4(j\omega + 2)}{(j\omega + 10)(j\omega + 100)}$$

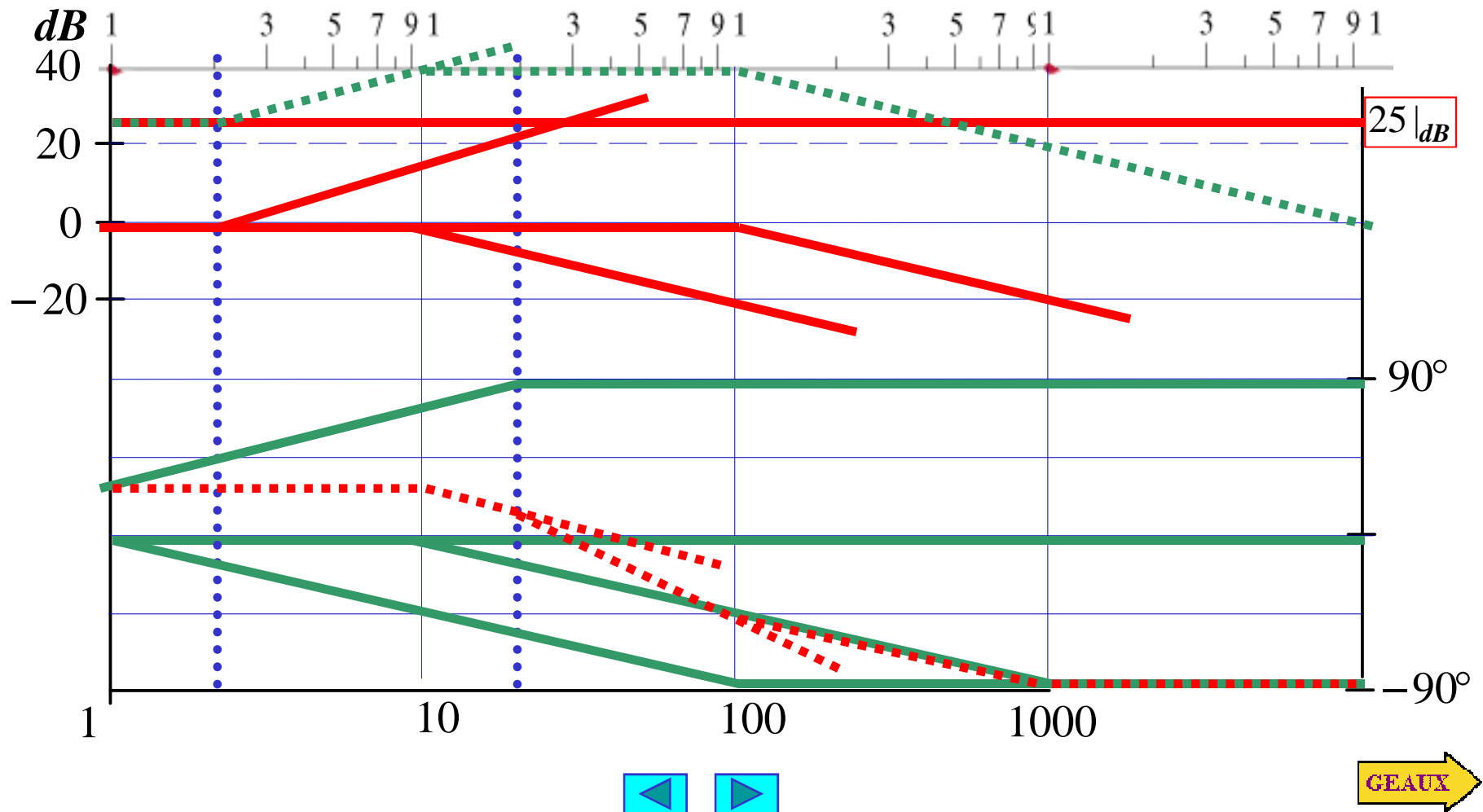
breaks: 2, 10, 100

But the function is NOT in standard form

Put in standard form

$$G(j\omega) = \frac{20(j\omega/2 + 1)}{(j\omega/10 + 1)(j\omega/100 + 1)}$$

We need to show about 4 decades

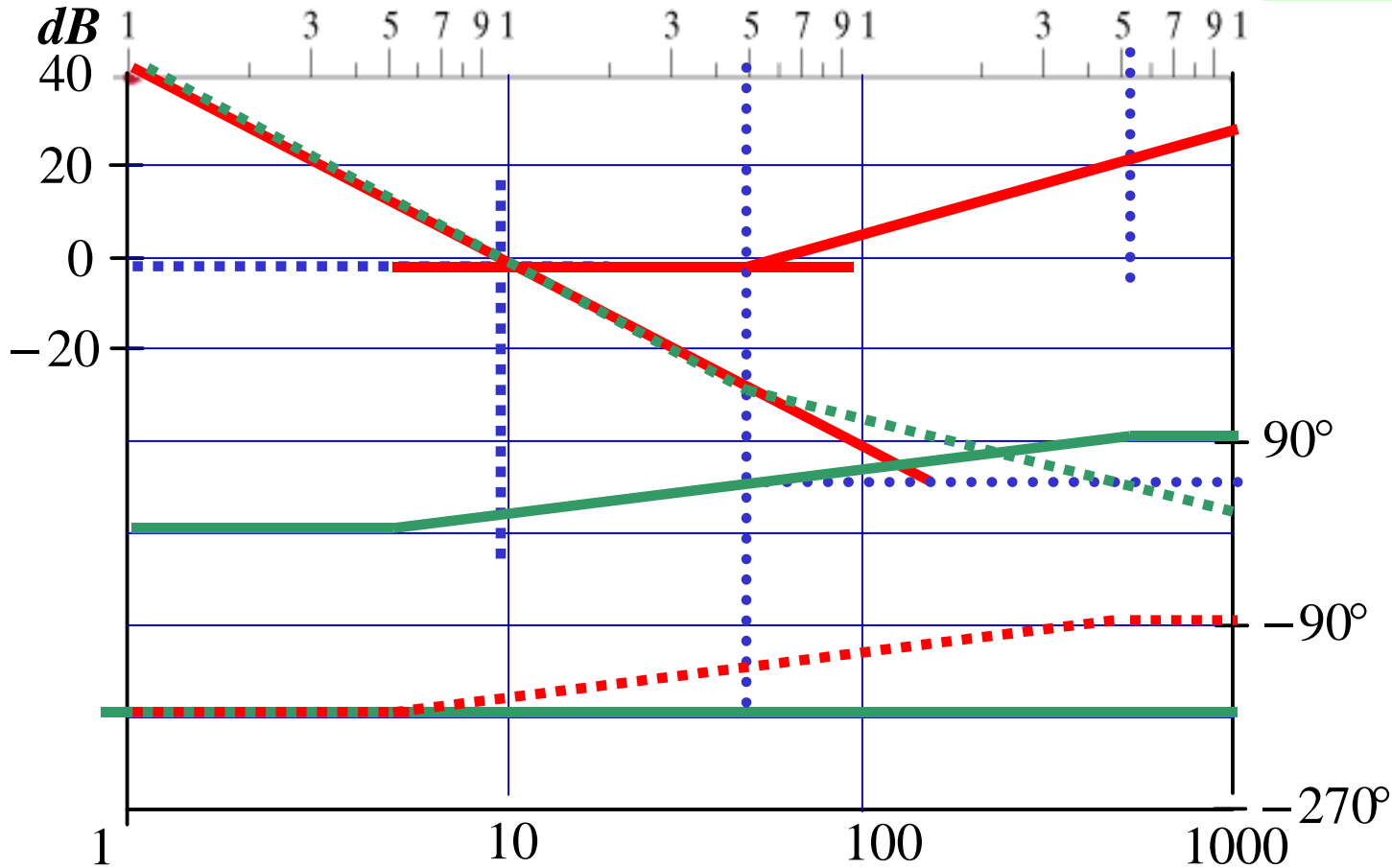


LEARNING EXTENSION

Sketch the magnitude characteristic

$$G(j\omega) = \frac{100(0.02j\omega + 1)}{(j\omega)^2}$$

It is in standard form
break at 50
Double pole at the origin



Once each term is drawn we form the composites



LEARNING EXTENSION

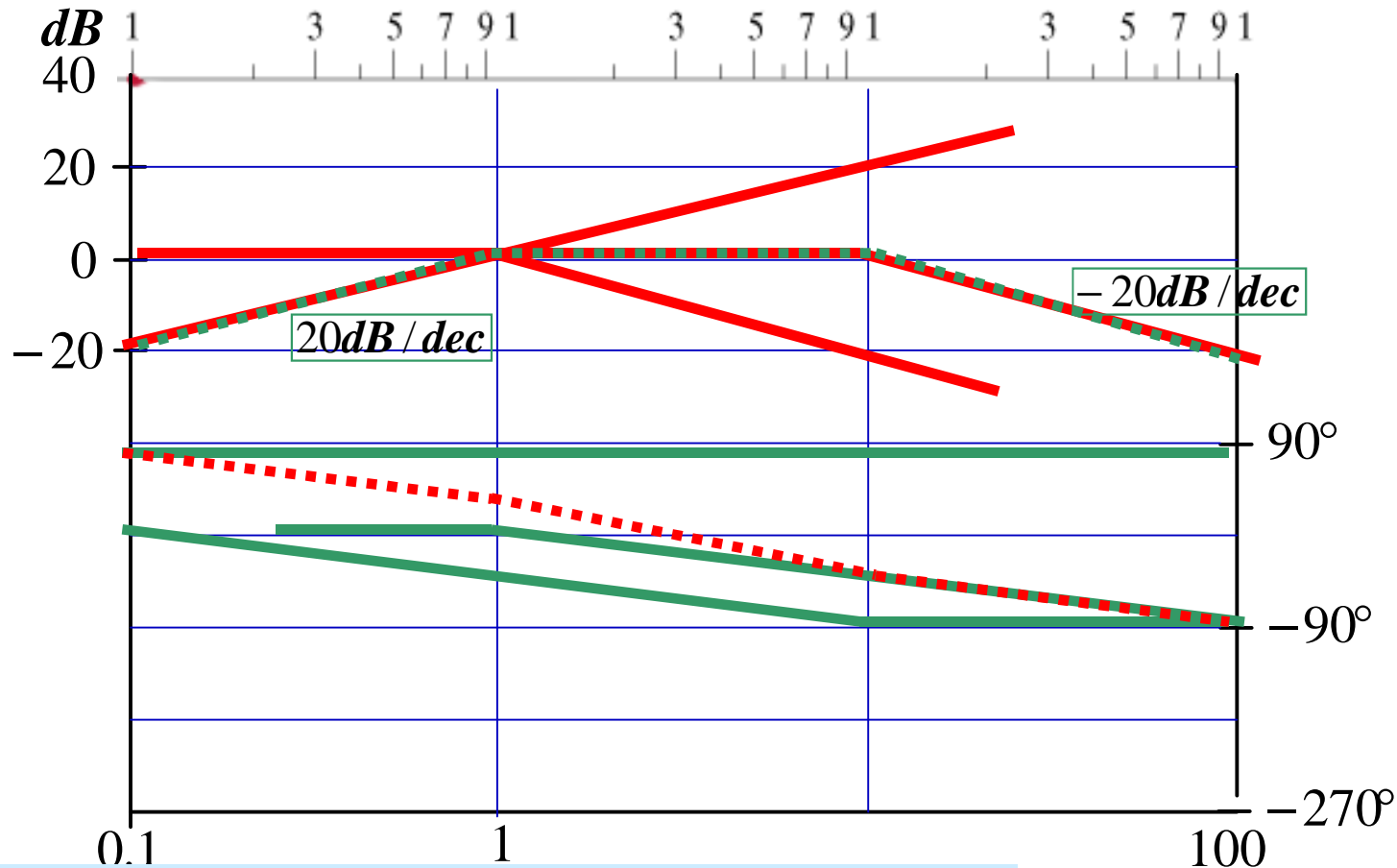
Put in standard form

$$G(j\omega) = \frac{j\omega}{(j\omega+1)(j\omega/10+1)}$$

Sketch the magnitude characteristic

$$G(j\omega) = \frac{10j\omega}{(j\omega+1)(j\omega+10)}$$

not in standard form
zero at the origin
breaks: 1, 10



Once each term is drawn we form the composites



LEARNING EXAMPLE A function with complex conjugate poles

$$t_2 = [1 + 2\zeta(j\omega\tau) + (j\omega\tau)^2]$$

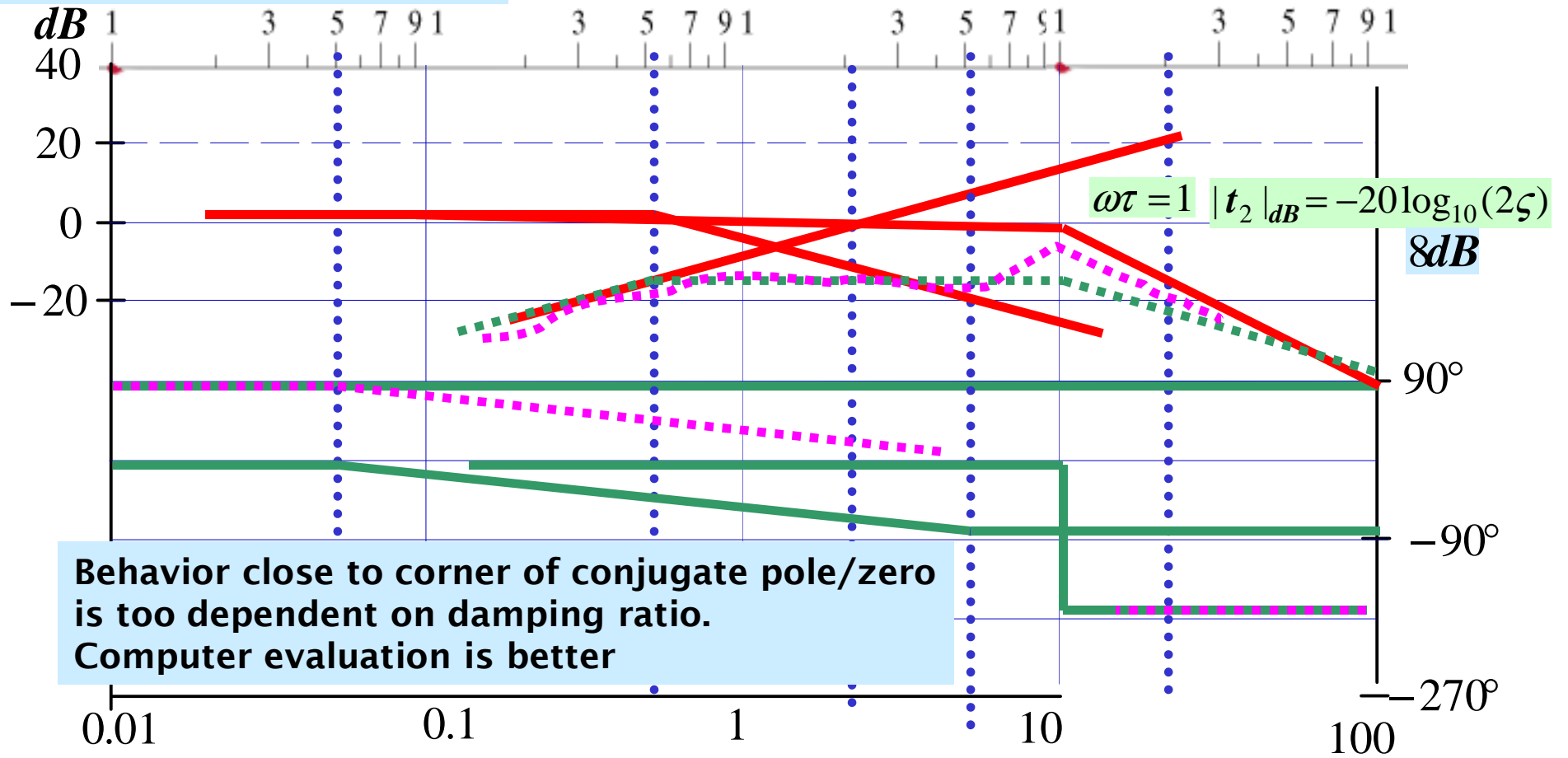
$$G(j\omega) = \frac{25j\omega}{(j\omega + 0.5)[(j\omega)^2 + 4j\omega + 100]}$$

$$\left. \begin{matrix} 2\zeta\tau = 1/25 \\ \tau = 0.1 \end{matrix} \right\} \Rightarrow \zeta = 0.2$$

Put in standard form

$$G(j\omega) = \frac{0.5j\omega}{(j\omega/0.5 + 1)[(j\omega/10)^2 + j\omega/25 + 1]}$$

Draw composite asymptote



Behavior close to corner of conjugate pole/zero is too dependent on damping ratio. Computer evaluation is better

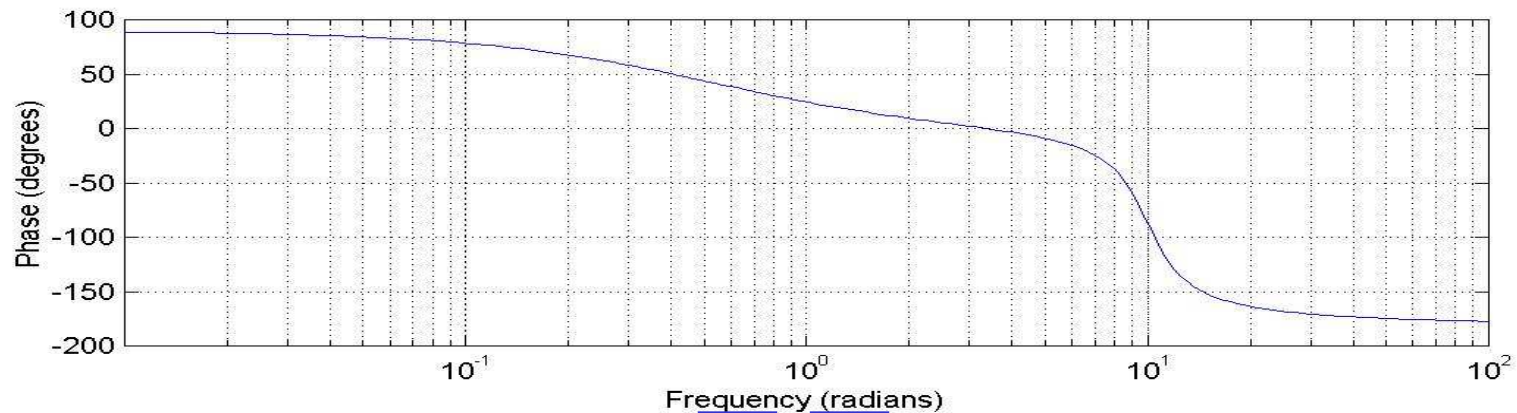
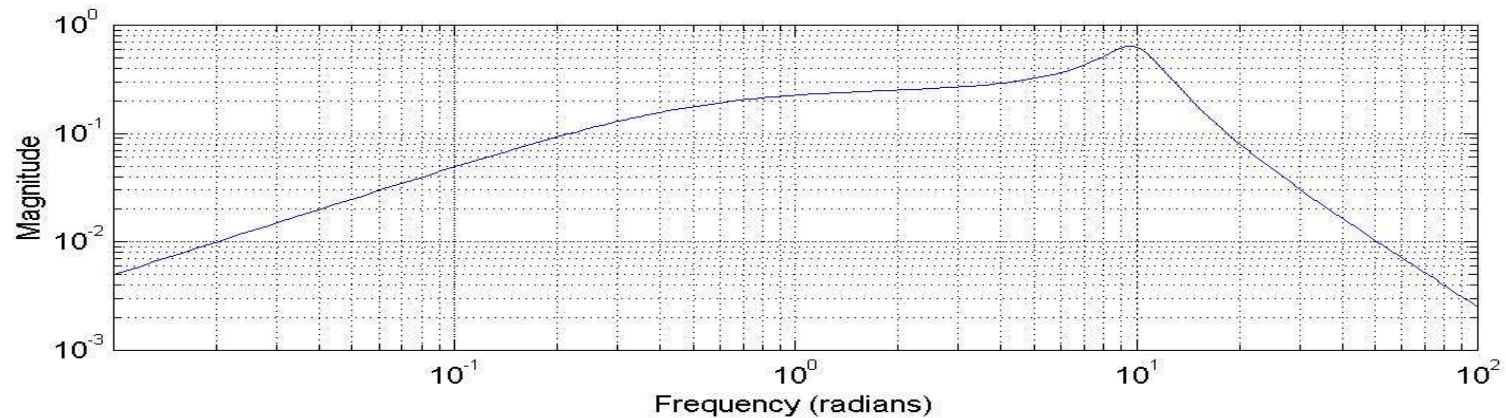


Evaluation of frequency response using MATLAB

Using default options

$$G(j\omega) = \frac{25j\omega}{(j\omega + 0.5)[(j\omega)^2 + 4j\omega + 100]}$$

```
» num=[25,0]; %define numerator polynomial
» den=conv([1,0.5],[1,4,100]) %use CONV for polynomial multiplication
den =
    1.0000    4.5000   102.0000    50.0000
» freqs(num,den)
```



Evaluation of frequency response using MATLAB User controlled

$$G(j\omega) = \frac{25j\omega}{(j\omega + 0.5)[(j\omega)^2 + 4j\omega + 100]}$$

```
>> clear all; close all %clear workspace and close any open figure
```

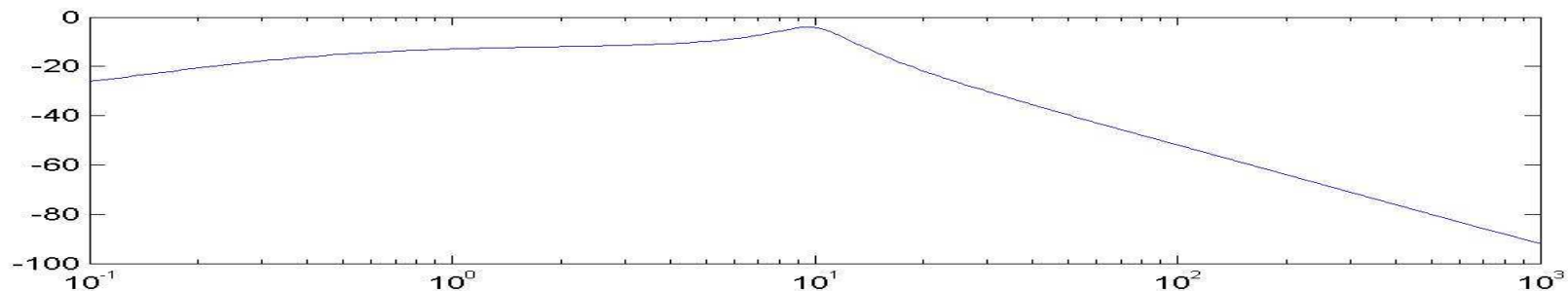
```
>> figure(1) %open one figure window (not STRICTLY necessary)
```

```
>> w=logspace(-1,3,200);%define x-axis, [10^{-1} - 10^3], 200pts total
```

```
>> G=25*j*w./((j*w+0.5).*(j*w).^2+4*j*w+100)); %compute transfer function
```

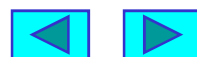
```
>> subplot(211) %divide figure in two. This is top part
```

```
>> semilogx(w,20*log10(abs(G))); %put magnitude here
```



```
>> grid %put a grid and give proper title and labels
```

```
>> ylabel('|G(j\omega)|(dB)'), title('Bode Plot: Magnitude response')
```

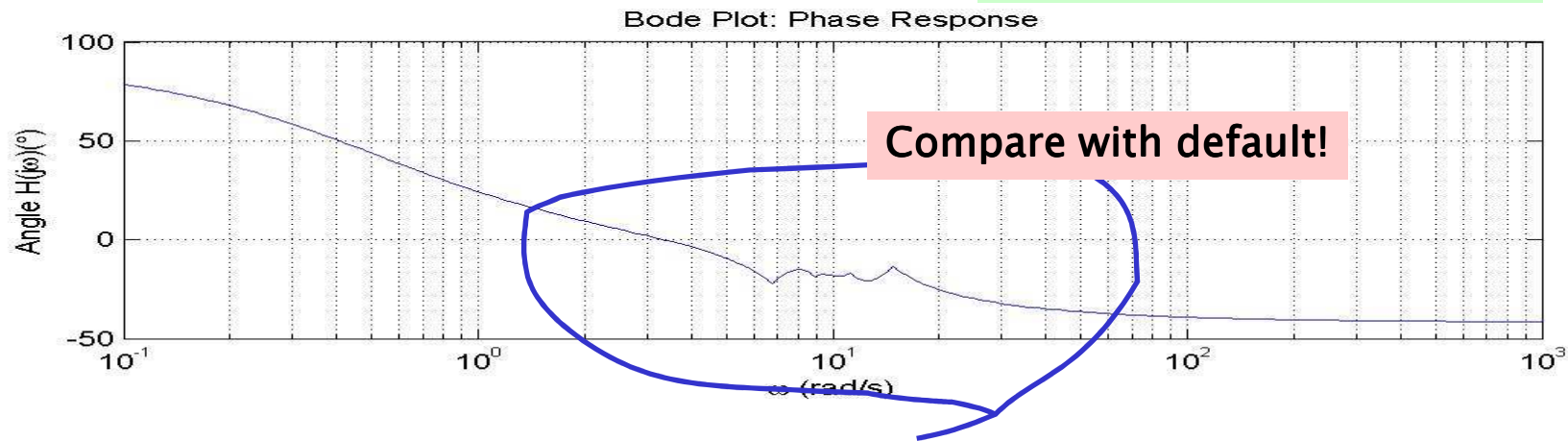
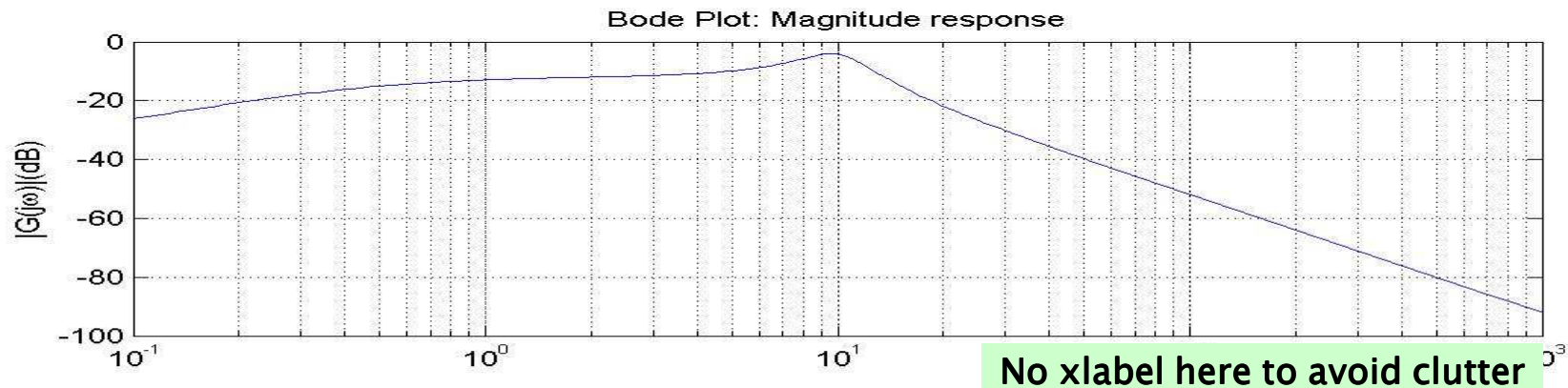


Evaluation of frequency response using MATLAB User controlled Continued

Repeat for phase

USE TO ZOOM IN A SPECIFIC REGION OF INTEREST

```
>> semilogx(w,unwrap(angle(G)*180/pi)) %unwrap avoids jumps from +180 to -180  
>> grid, ylabel('Angle H(j\omega)(\circ)'), xlabel('\omega (rad/s)')  
>> title('Bode Plot: Phase Response')
```



LEARNING EXTENSION

Sketch the magnitude characteristic

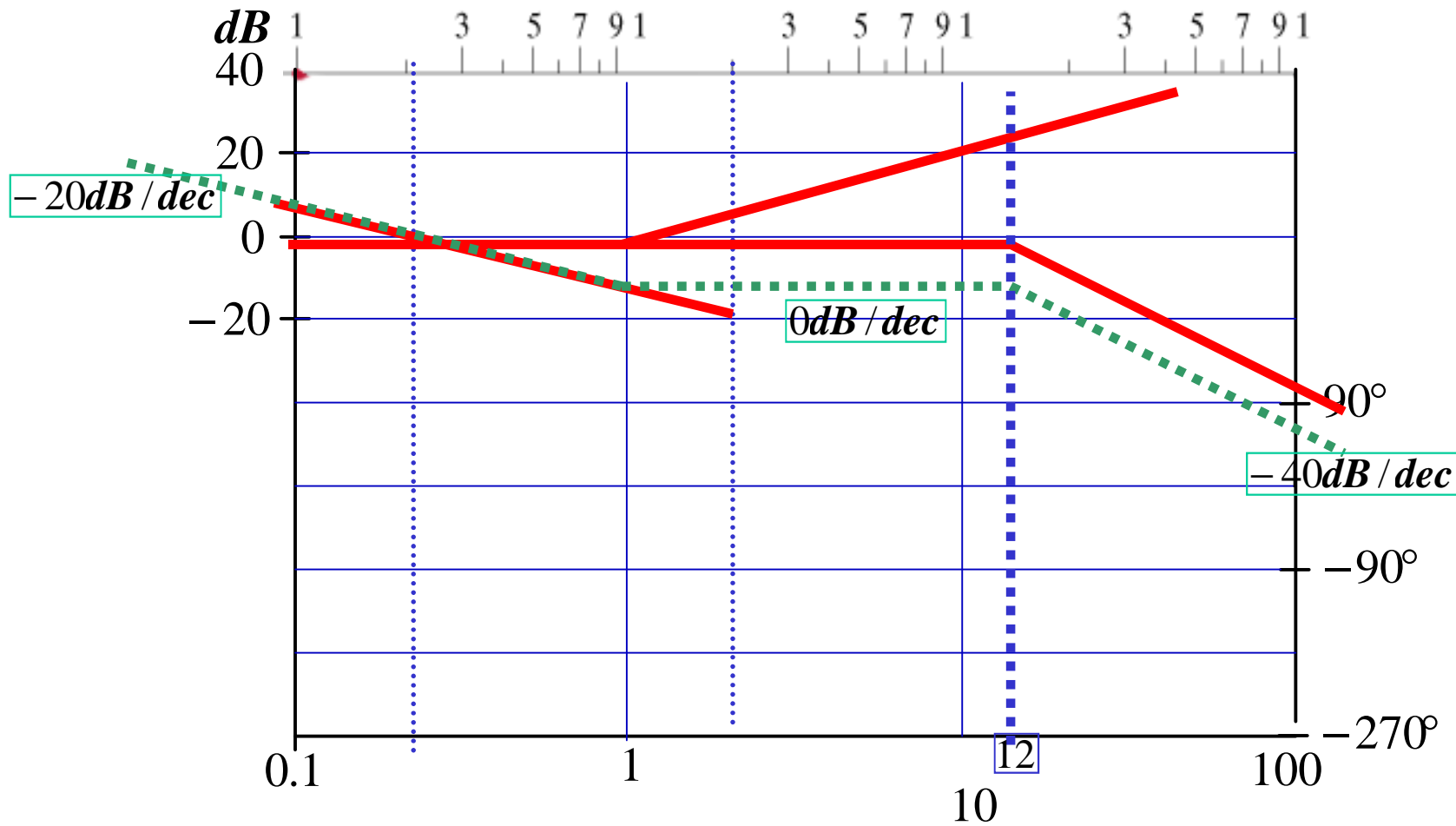
$$t_2 = [1 + 2\zeta(j\omega\tau) + (j\omega\tau)^2]$$

$$G(j\omega) = \frac{0.2(j\omega + 1)}{j\omega[(j\omega/12)^2 + j\omega/36 + 1]}$$

$$\tau = 1/12$$

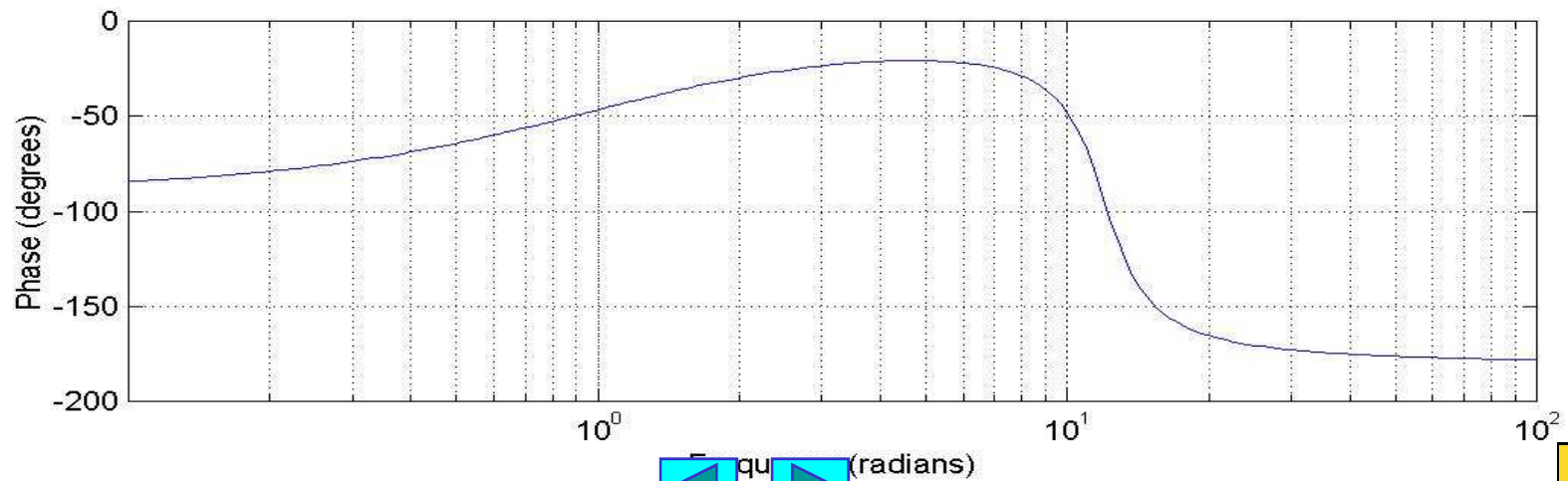
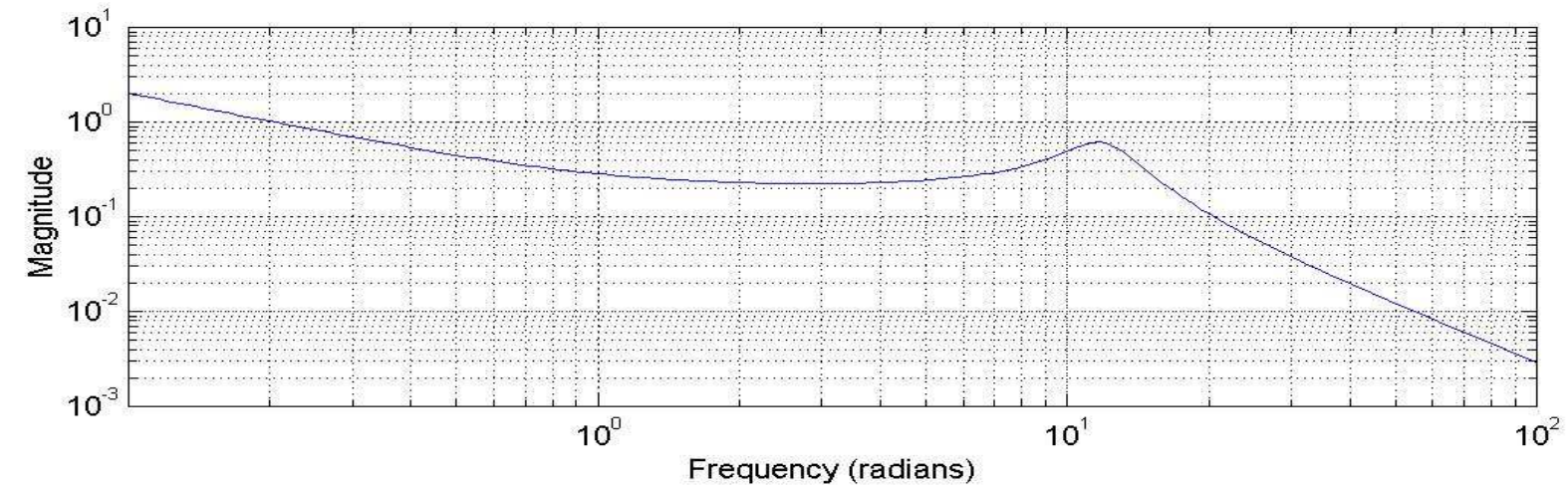
$$2\zeta\tau = 1/36 \Rightarrow \zeta = 1/6$$

$$\omega\tau = 1 \quad |t_2|_{dB} = -20\log_{10}(2\zeta) = 9.5dB$$



$$G(j\omega) = \frac{0.2(j\omega + 1)}{j\omega[(j\omega/12)^2 + j\omega/36 + 1]}$$

```
» num=0.2*[1,1];  
» den=conv([1,0],[1/144,1/36,1]);  
» freqs(num,den)
```

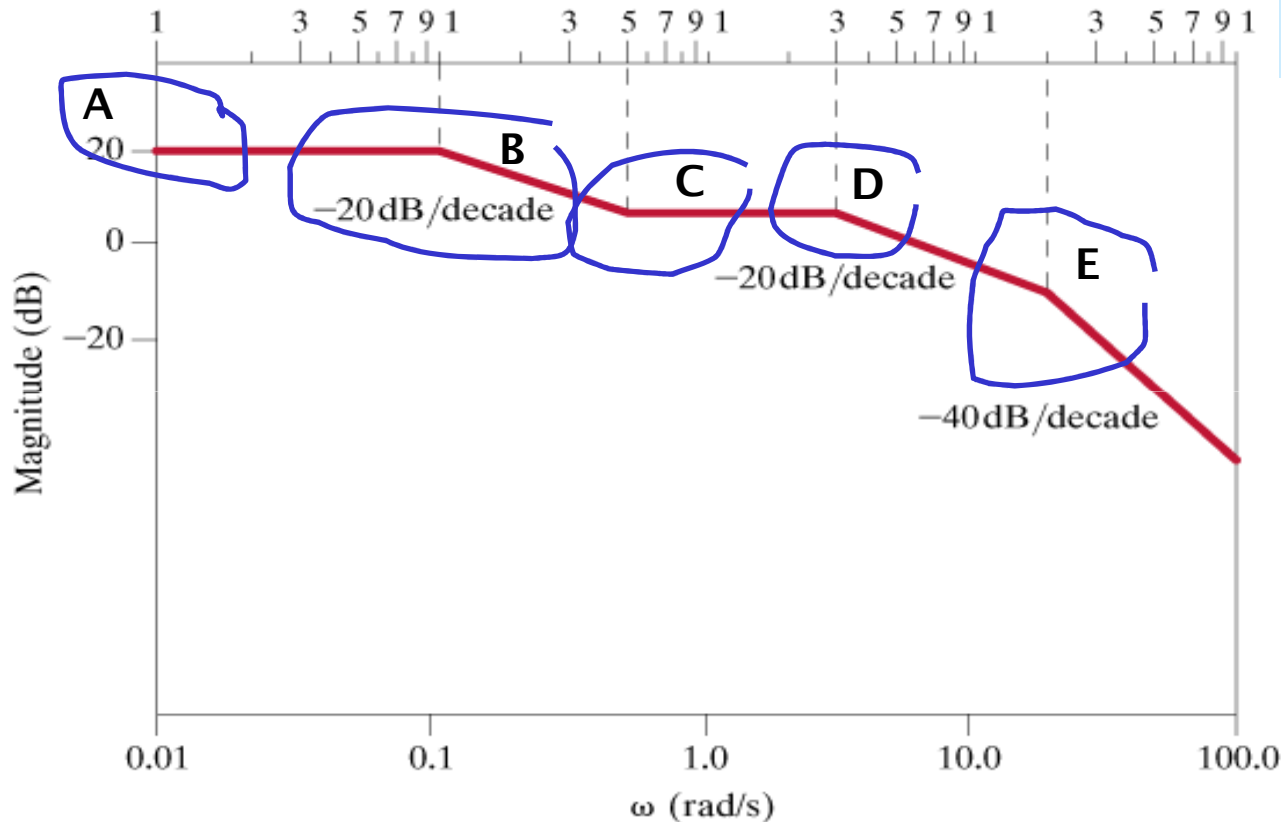


← qu → (radians)



DETERMINING THE TRANSFER FUNCTION FROM THE BODE PLOT

This is the inverse problem of determining frequency characteristics. We will use only the composite asymptotes plot of the magnitude to postulate a transfer function. The slopes will provide information on the order



A. different from 0dB.
There is a constant K_0

$$K_0 |_{dB} = 20 \Rightarrow K_0 = 10^{\frac{K_0 |_{dB}}{20}}$$

B. Simple pole at 0.1

$$(j\omega/0.1 + 1)^{-1}$$

C. Simple zero at 0.5

$$(j\omega/0.5 + 1)$$

D. Simple pole at 3

$$(j\omega/3 + 1)^{-1}$$

E. Simple pole at 20

$$(j\omega/20 + 1)^{-1}$$

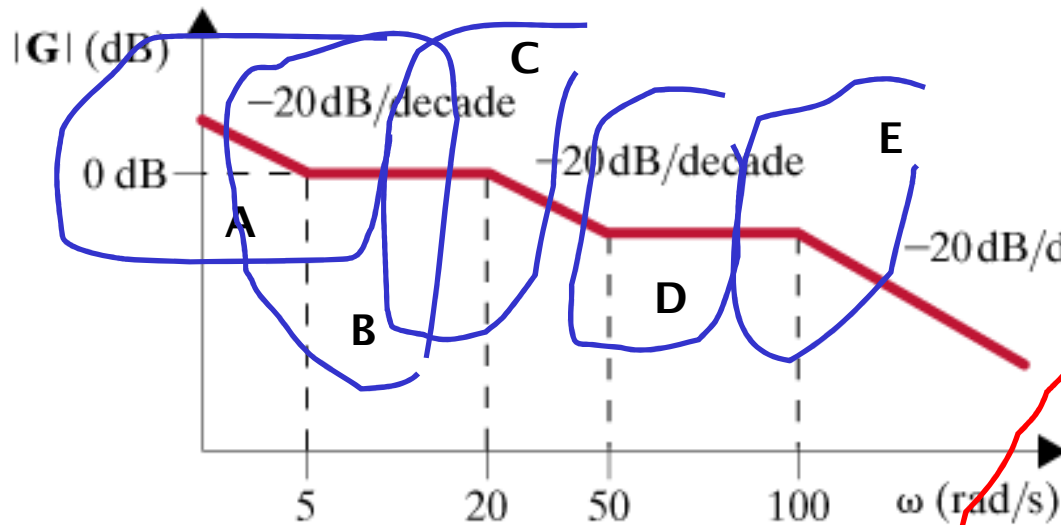
$$G(j\omega) = \frac{10(j\omega/0.5 + 1)}{(j\omega/0.1 + 1)(j\omega/3 + 1)(j\omega/20 + 1)}$$

If the slope is -40dB we assume double real pole. Unless we are given more data



LEARNING EXTENSION

Determine a transfer function from the composite magnitude asymptotes plot



A. Pole at the origin.
Crosses 0dB line at 5

$$\frac{5}{j\omega}$$

B. Zero at 5

C. Pole at 20

D. Zero at 50

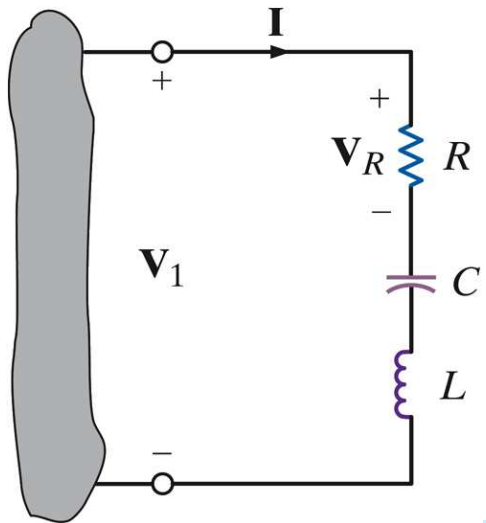
E. Pole at 100

$$G(j\omega) = \frac{5(j\omega/5 + 1)(j\omega/50 + 1)}{j\omega(j\omega/20 + 1)(j\omega/100 + 1)}$$

Sinusoidal



RESONANT CIRCUITS – SERIES RESONANCE



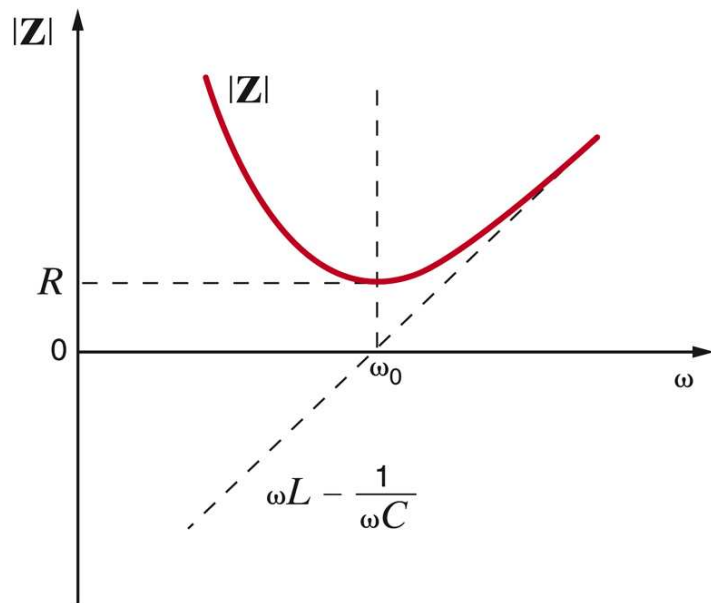
$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$\omega L = \frac{1}{\omega C} \Rightarrow \text{Im} \{ Z \} = 0$$

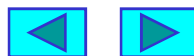
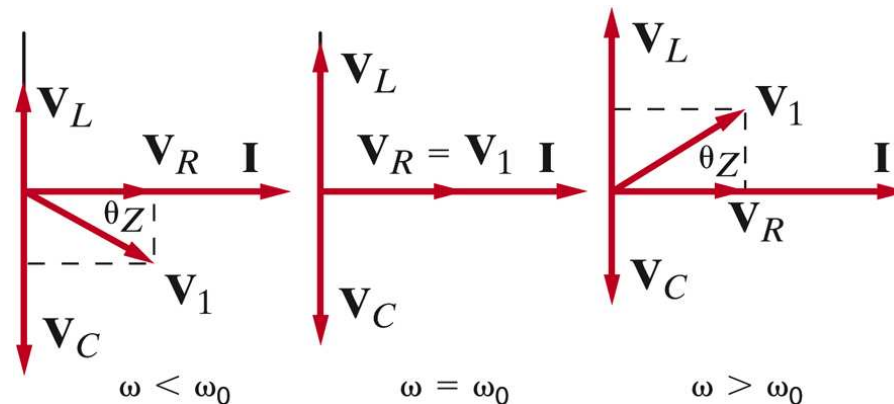
$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow Z(j\omega_0) = R \quad Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R} = \frac{1}{R} \sqrt{\frac{L}{C}}$$

RESONANT FREQUENCY

QUALITY FACTOR

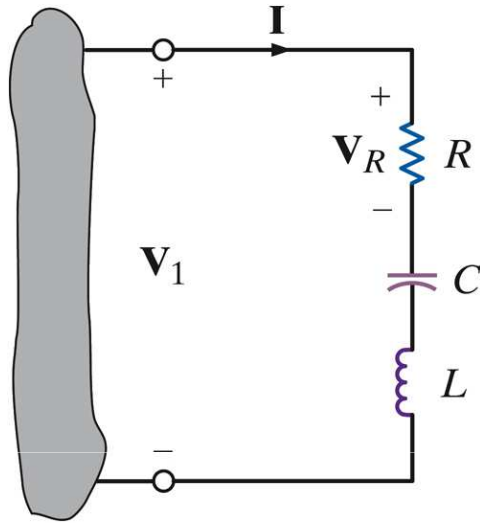


PHASOR DIAGRAM



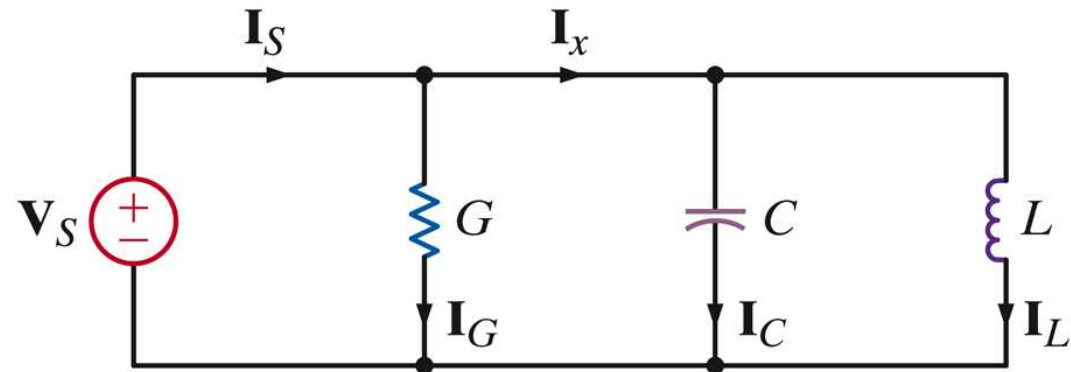
RESONANT CIRCUITS

These are circuits with very special frequency characteristics...
And resonance is a very important physical phenomenon



Series RLC circuit

$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$



Parallel RLC circuit

$$Y(j\omega) = G + j\omega C + \frac{1}{j\omega L}$$

The reactance of each circuit is zero when

$$\omega L = \frac{1}{\omega C} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

The frequency at which the circuit becomes purely resistive is called the resonance frequency

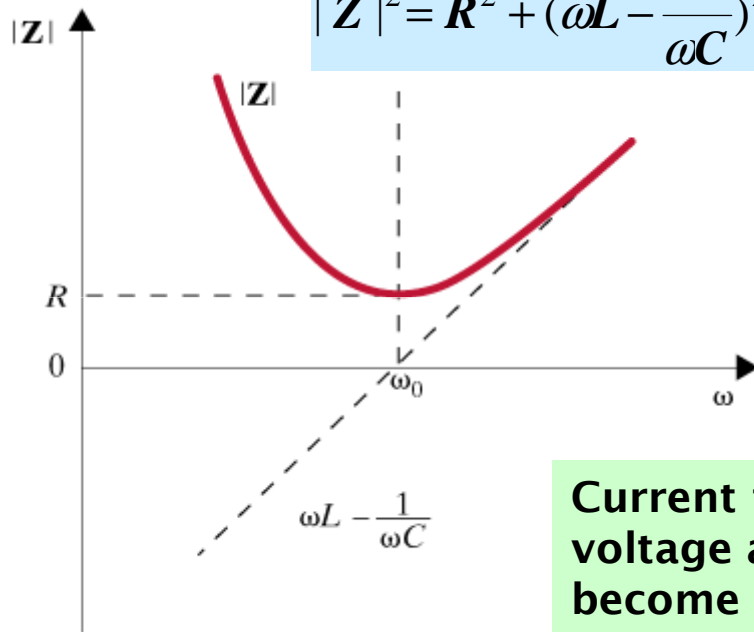


Properties of resonant circuits

At resonance the impedance/admittance is minimal

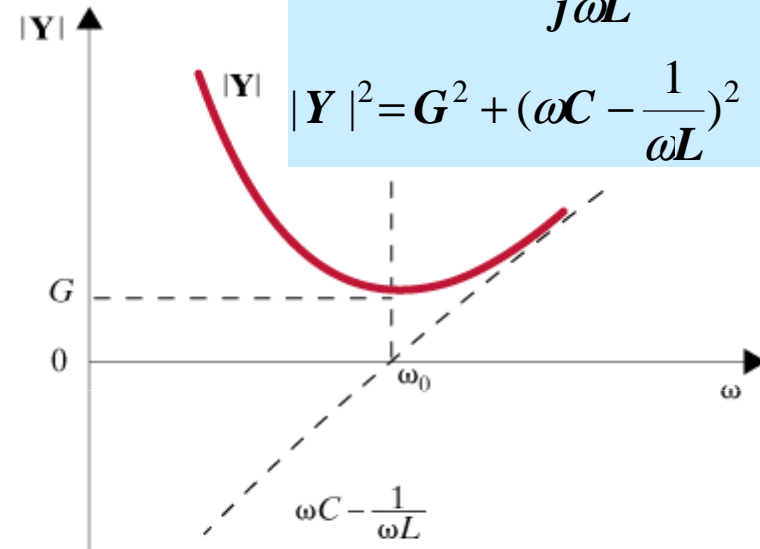
$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

$$|Z|^2 = R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2$$



$$Y(j\omega) = G + \frac{1}{j\omega L} + j\omega C$$

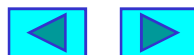
$$|Y|^2 = G^2 + \left(\omega C - \frac{1}{\omega L}\right)^2$$



Current through the serial circuit/
voltage across the parallel circuit can
become very large (if resistance is small)

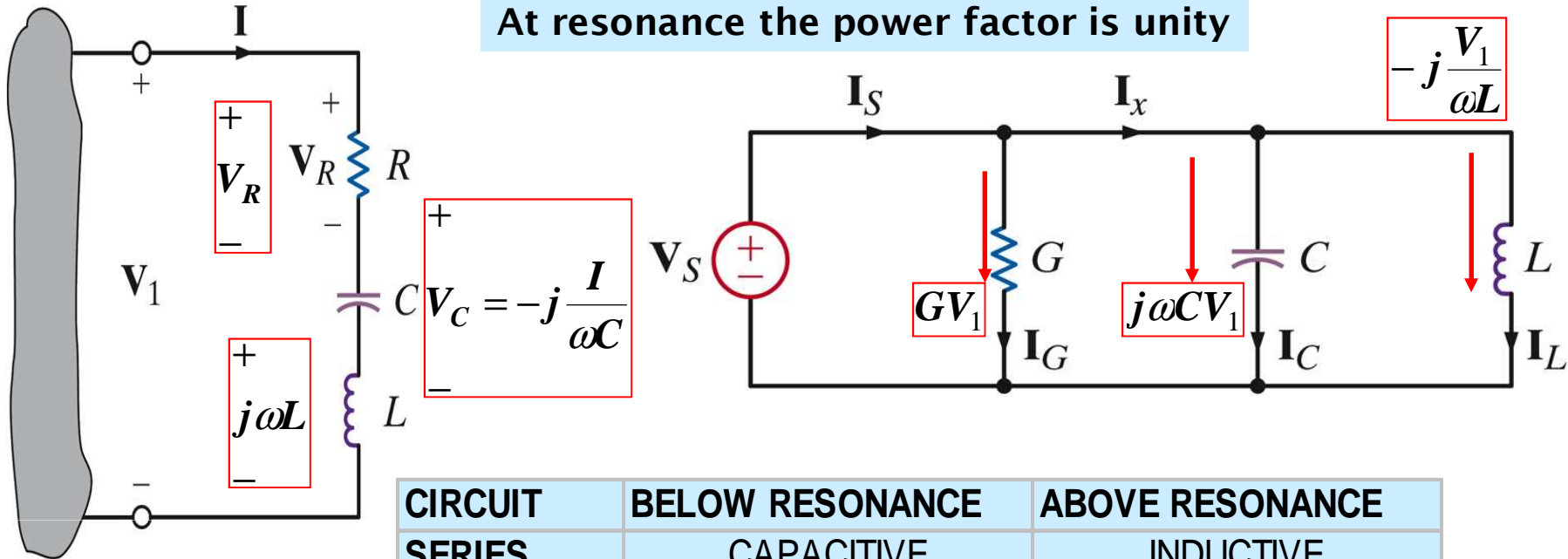
$$\text{Quality Factor: } Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}$$

Given the similarities between series and parallel resonant circuits,
we will focus on serial circuits



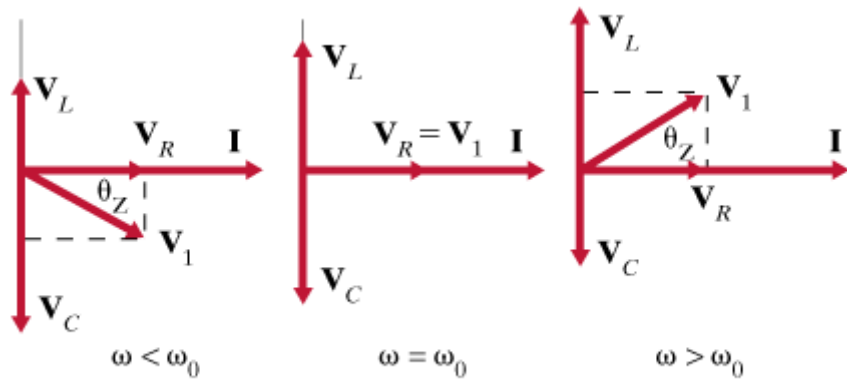
Properties of resonant circuits

At resonance the power factor is unity

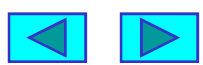
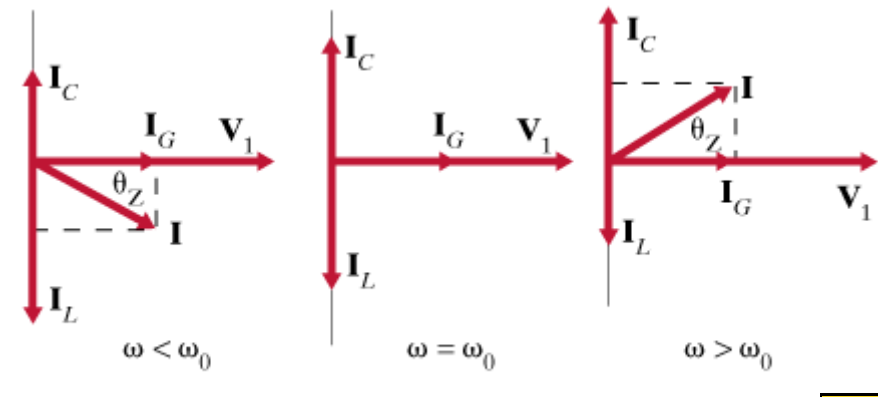


CIRCUIT	BELOW RESONANCE	ABOVE RESONANCE
SERIES	CAPACITIVE	INDUCTIVE
PARALLEL	INDUCTIVE	CAPACITIVE

Phasor diagram for series circuit

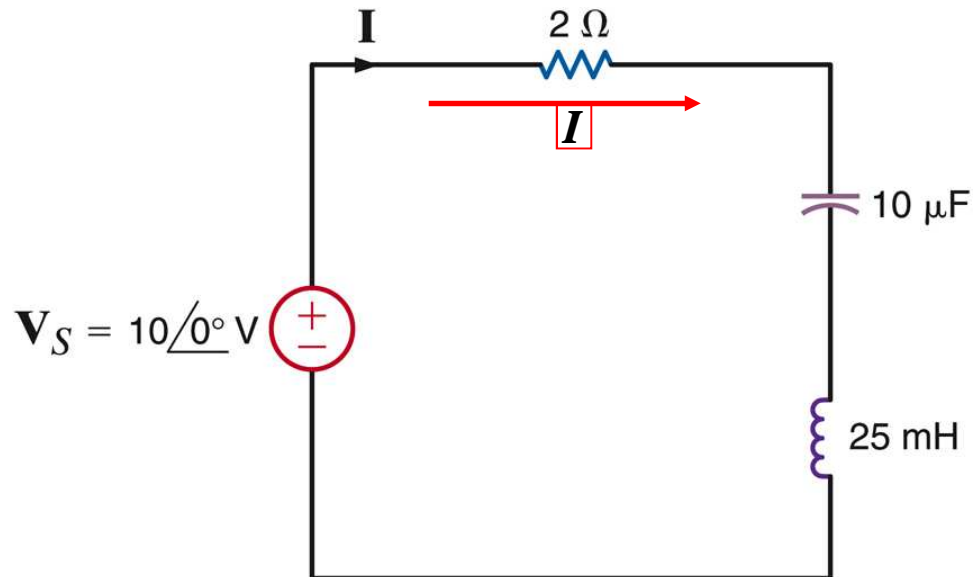


Phasor diagram for parallel circuit



LEARNING EXAMPLE

Determine the resonant frequency, the voltage across each element at resonance and the value of the quality factor



$$\frac{1}{\omega_0 C} = \omega_0 L = 50 \Omega$$

$$V_C = \frac{1}{j\omega_0 C} I = -j50 \times 5 = 250 \angle -90^\circ$$

$$Q = \frac{\omega_0 L}{R} = \frac{50}{2} = 25$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(25 \times 10^{-3} \text{ H})(10 \times 10^{-6} \text{ F})}} = 2000 \text{ rad/sec}$$

At resonance $Z = 2 \Omega$

$$I = \frac{V_S}{Z} = \frac{10 \angle 0^\circ}{2} = 5 \text{ A}$$

$$\omega_0 L = (2 \times 10^3)(25 \times 10^{-3}) = 50 \Omega$$

$$V_L = j\omega_0 L I = j50 \times 5 = 250 \angle 90^\circ \text{ (V)}$$

At resonance

$$|V_L| = \omega_0 L \left| \frac{V_S}{R} \right| = Q |V_S|$$

$$|V_C| = Q |V_S|$$



LEARNING EXAMPLE

Given $L = 0.02\text{H}$ with a Q factor of 200, determine the capacitor necessary to form a circuit resonant at 1000Hz

$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow 2\pi \times 1000 = \frac{1}{\sqrt{0.02C}} \Rightarrow C = 1.27 \mu\text{F}$$

What is the rating for the capacitor if the circuit is tested with a 10V supply?

At resonance

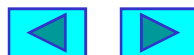
$$|V_L| = \omega_0 L \left| \frac{V_S}{R} \right| = Q |V_S|$$

$$|V_C| = Q |V_S| \Rightarrow |V_C| = 2000\text{V}$$

$$L \text{ with } Q = 200 \Rightarrow 200 = \frac{\omega_0 L}{R} \Rightarrow R = \frac{2\pi \times 1000 \times 0.02}{200} = 1.59\Omega$$

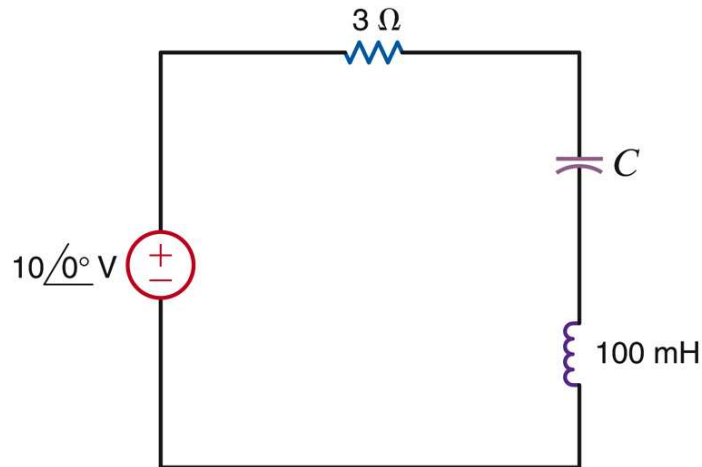
$$I = \frac{10}{1.59} = 6.28\text{A}$$

The reactive power on the capacitor exceeds 12kVA



LEARNING EXTENSION

Find the value of C that will place the circuit in resonance at 1800rad/sec



$$\omega_0 = \frac{1}{\sqrt{LC}} \quad 1800 = \frac{1}{\sqrt{0.1(H) \times C}} \Rightarrow C = \frac{1}{0.1 \times 1800^2}$$

$$C = 3.86 \mu F$$

Find the Q for the network and the magnitude of the voltage across the capacitor

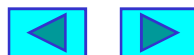
$$Q = \frac{\omega_0 L}{R} \quad Q = \frac{1800 \times 0.1}{3} = 60$$

At resonance

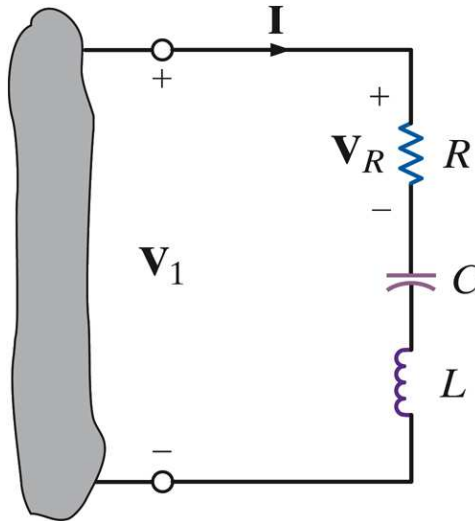
$$|V_L| = \omega_0 L \left| \frac{V_S}{R} \right| = Q |V_S|$$

$$|V_C| = 600V$$

$$|V_C| = Q |V_S|$$



Resonance for the series circuit



$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

$$|Z|^2 = R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2$$

Claim: The voltage gain

$$G_v = \frac{V_R}{V_1} = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

$$G_v = \frac{R}{R + j\omega L + \frac{1}{j\omega C}} = \frac{R}{Z(j\omega)}$$

At resonance:

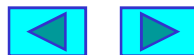
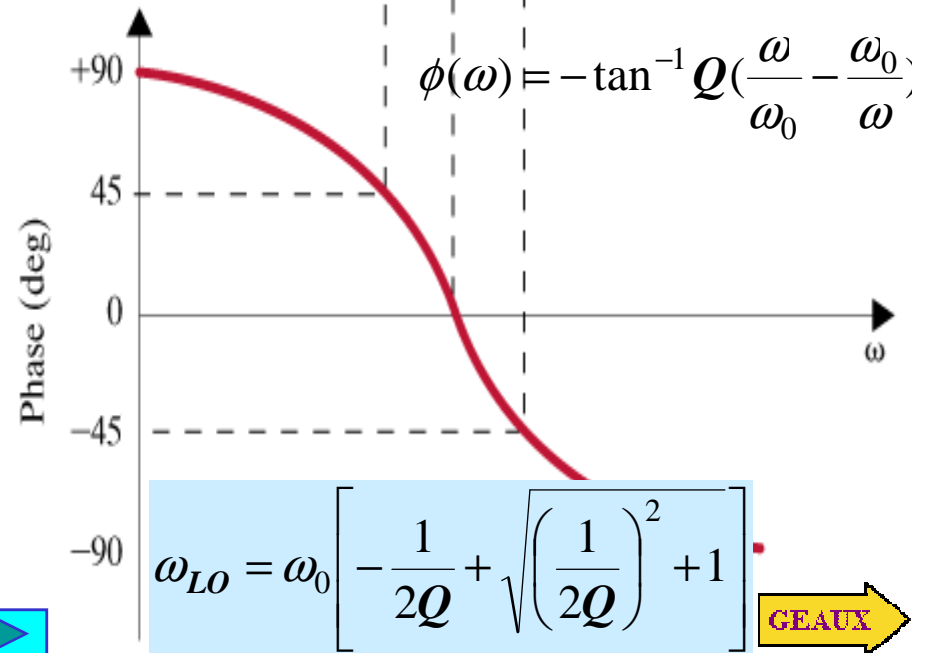
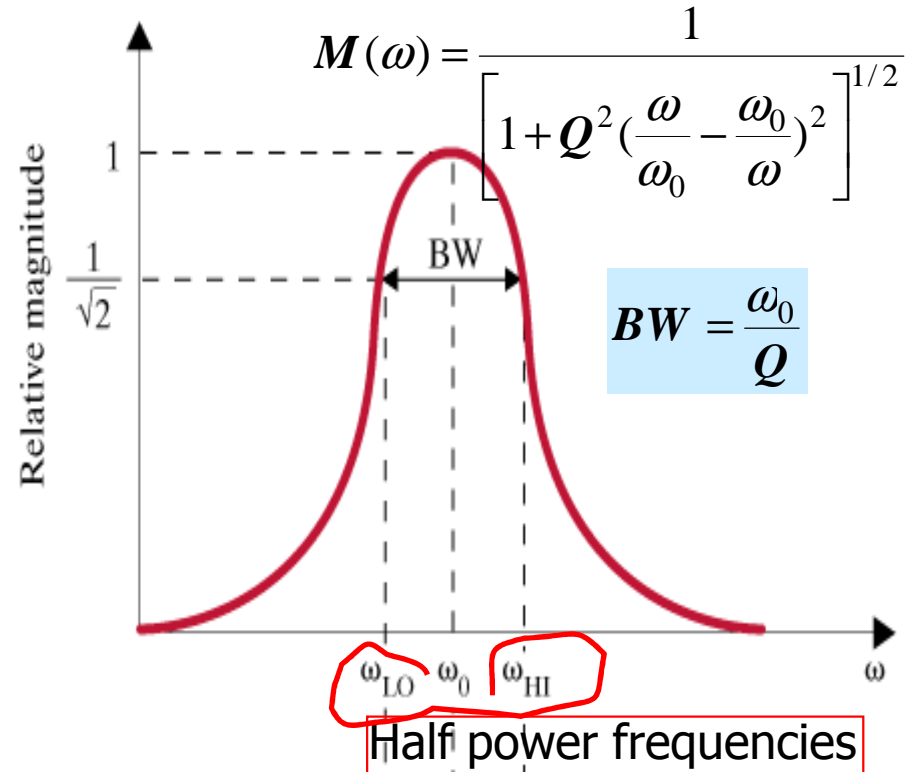
$$\omega_0 L = QR, \quad \omega_0 C = \frac{1}{QR}$$

$$Z(j\omega) = R + j\frac{\omega}{\omega_0}QR - j\frac{\omega_0}{\omega}QR$$

$$= R \left[1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \right]$$

$$G_v = \frac{R}{Z}$$

$$M(\omega) = |G_v|, \quad \phi(\omega) = \angle G_v$$



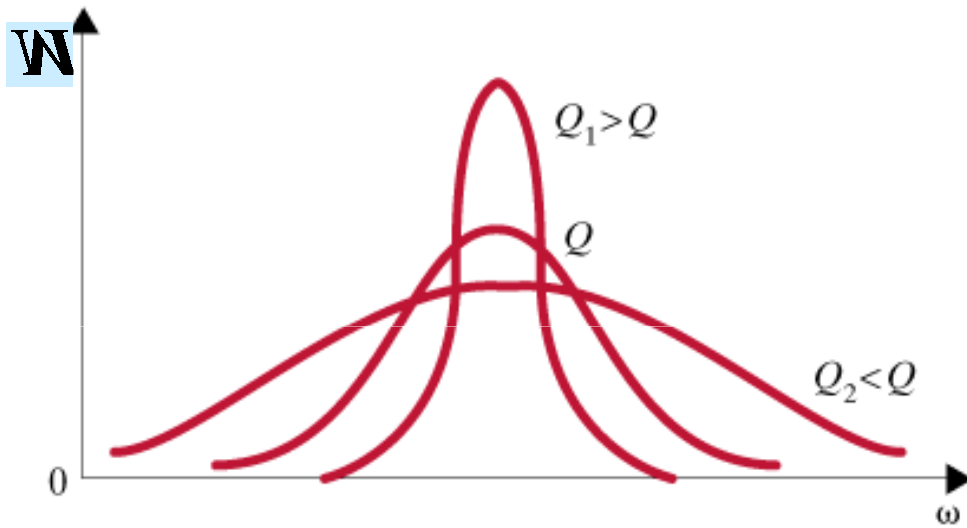
The Q factor

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

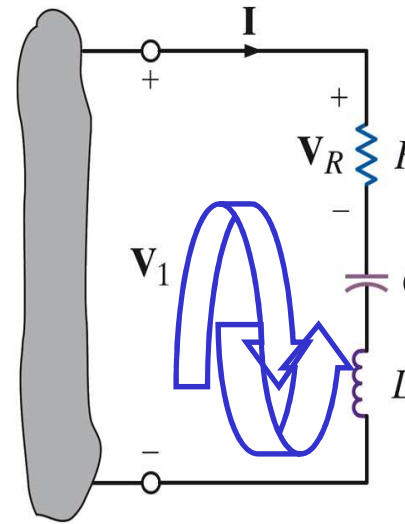
For series circuit: High Q \Leftrightarrow Low R

For parallel circuit: High Q \Leftrightarrow High R (low G)

High Q \Leftrightarrow Small BW



Q can also be interpreted from an energy point of view



dissipates

Stores as E field

Stores as M field

Capacitor and inductor exchange stored energy. When one is at maximum the other is at zero

$$Q = 2\pi \frac{W_S}{W_D} = 2\pi \frac{\text{maximum energy stored}}{\text{energy dissipated by cycle}}$$

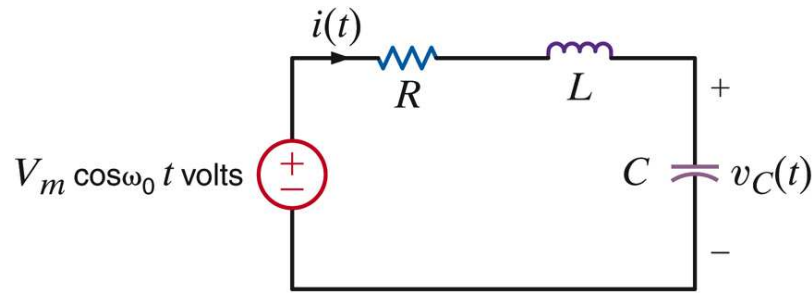
$$W_D = RI_{eff}^2 \times \frac{\omega_0}{2\pi} = \frac{1}{2} RI_{mx}^2 \times \frac{\omega_0}{2\pi}$$

$$W_S = \frac{1}{2} LI_{mx}^2 = \frac{1}{2} CV_{mx}^2$$

$$\frac{W_s}{W_D} = \frac{L\omega_0}{2\pi \times R} = \frac{Q}{2\pi}$$



ENERGY TRANSFER IN RESONANT CIRCUITS



$$w_L(t) + w_C(t) = \frac{V_m^2 L}{2R^2} (\cos^2 \omega_0 t + \sin^2 \omega_0 t)$$

$$\mathbf{V}_C = \frac{1}{j\omega_0 C} \mathbf{I} = \frac{1}{j\omega_0 C} \frac{V_m}{R} \angle 0^\circ = \frac{V_m}{\omega_0 RC} \angle -90^\circ$$

$$v_C(t) = \frac{V_m}{\omega_0 RC} \cos(\omega_0 t - 90^\circ) = \frac{V_m}{\omega_0 RC} \sin \omega_0 t \text{ volts}$$

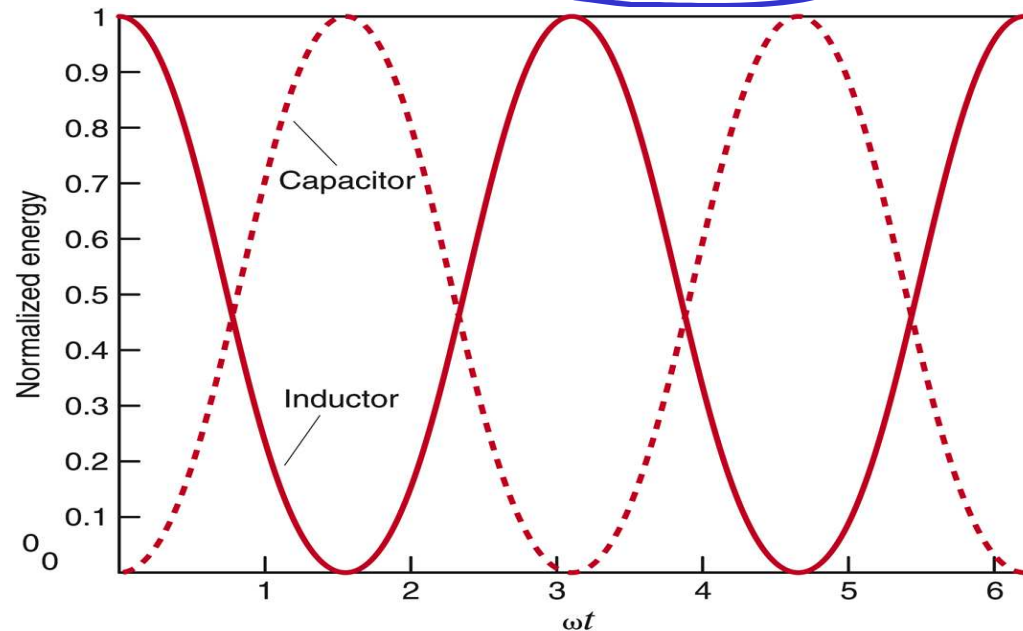
$$i(t) = \frac{V_m}{R} \cos \omega_0 t [A]$$

$$w_C(t) = \frac{1}{2} C v_C^2(t) = \frac{1}{2} C \left(\frac{V_m}{\omega_0 RC} \sin \omega_0 t \right)^2 = \frac{V_m^2}{2\omega_0^2 R^2 C} \sin^2 \omega_0 t \text{ joules}$$

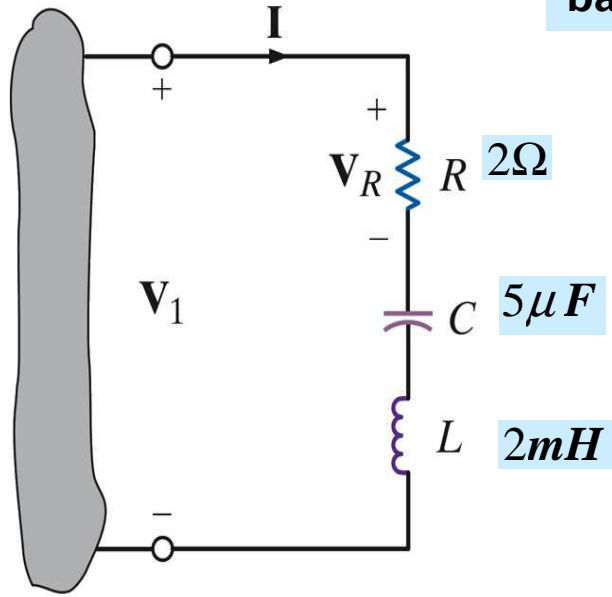
$$w_L(t) = \frac{1}{2} L i^2(t) = \frac{1}{2} L \left(\frac{V_m}{R} \cos \omega_0 t \right)^2 = \frac{V_m^2 L}{2R^2} \cos^2 \omega_0 t \text{ joules}$$

$$\frac{V_m^2 L}{2R^2}$$

Normalization factor



LEARNING EXAMPLE



Determine the resonant frequency, quality factor and bandwidth when $R=2$ and when $R=0.2$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}$$

$$BW = \frac{\omega_0}{Q}$$

$$\omega_0 = \frac{1}{\sqrt{(2 \times 10^{-3})(5 \times 10^{-6})}} = 10^4 \text{ rad/sec}$$

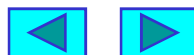
R	Q
2	10
0.2	100

R	Q	BW(rad/sec)
2	10	1000
0.2	100	100

$$Q = \frac{10000 \times 0.002}{R}$$

$$BW = 10000 / Q$$

Evaluated with EXCEL



LEARNING EXTENSION

A series RLC circuit as the following properties:

$$R = 4\Omega, \omega_0 = 4000\text{rad/sec}, BW = 100\text{rad/sec}$$

Determine the values of L, C.

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

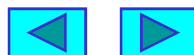
$$BW = \frac{\omega_0}{Q}$$

1. Given resonant frequency and bandwidth determine Q.
2. Given R, resonant frequency and Q determine L, C.

$$Q = \frac{\omega_0}{BW} = \frac{4000}{100} = 40$$

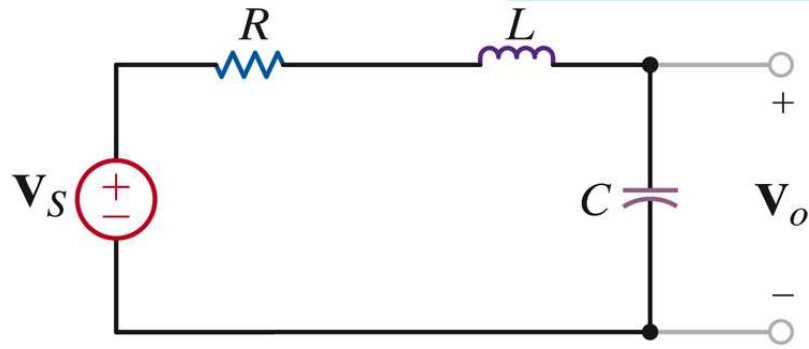
$$L = \frac{QR}{\omega_0} = \frac{40 \times 4}{4000} = 0.040\text{H}$$

$$C = \frac{1}{L\omega_0^2} = \frac{1}{\omega_0 R Q} = \frac{1}{4 \times 10^{-2} \times 16 \times 10^6} = 1.56 \times 10^{-6}\text{F}$$



LEARNING EXAMPLE

Find R, L, C so that the circuit operates as a band-pass filter with center frequency of 1000rad/s and bandwidth of 100rad/s



$$G_v = \frac{R}{R + j\omega L + \frac{1}{j\omega C}} = \frac{R}{Z(j\omega)}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

$$BW = \frac{\omega_0}{Q}$$

dependent

Strategy:

1. Determine Q
2. Use value of resonant frequency and Q to set up two equations in the three unknowns
3. Assign a value to one of the unknowns

$$Q = \frac{\omega_0}{BW} = \frac{1000}{100} = 10$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow (10^3)^2 = \frac{1}{LC}$$

$$Q = \frac{\omega_0 L}{R} \Rightarrow 10 = \frac{1000L}{R}$$

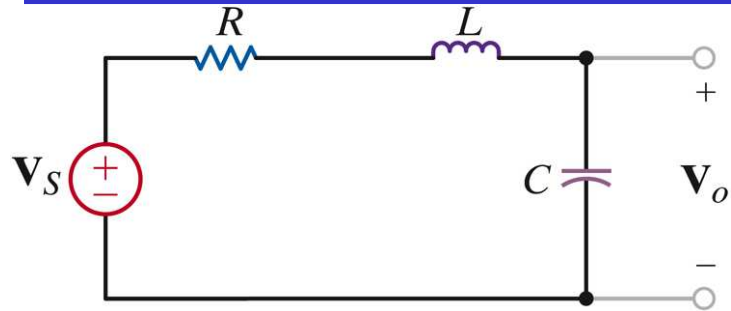
For example $C = 1\mu F = 10^{-6} F$

$$L = 1H$$

$$R = 100\Omega$$



PROPERTIES OF RESONANT CIRCUITS: VOLTAGE ACROSS CAPACITOR



At resonance

$$|V_o| = Q |V_R|$$

But this is NOT the maximum value for the voltage across the capacitor

$$\left| \frac{V_o}{V_S} \right| = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{1 - \omega^2 LC + j\omega CR}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

$$u = \frac{\omega}{\omega_0}; g = \left| \frac{V_o}{V_S} \right|^2$$

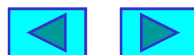
$$g(u) = \frac{1}{\left[(1 - u^2)^2 + \left(\frac{u}{Q} \right)^2 \right]}$$

$$\frac{dg}{du} = 0 = \frac{2(1 - u^2)(-2u) + 2(u/Q)(1/Q)}{\left[(1 - u^2)^2 + \left(\frac{u}{Q} \right)^2 \right]^2} \Rightarrow 2(1 - u^2) = \frac{1}{Q^2}$$

$$u_{\max} = \frac{\omega_{\max}}{\omega_0} = \sqrt{1 - \frac{1}{2Q^2}}$$

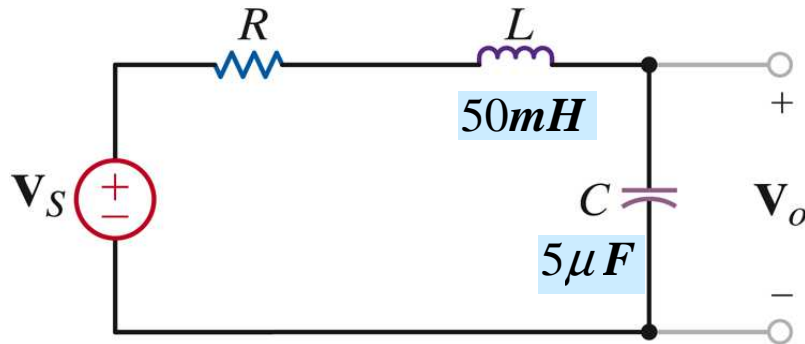
$$g_{\max} = \frac{1}{\frac{1}{4Q^4} + \left(\frac{1}{Q^2} - \frac{1}{2Q^4} \right)} = \frac{Q^2}{1 - \frac{1}{4Q^2}}$$

$$|V_o| = \frac{Q |V_S|}{\sqrt{1 - \frac{1}{4Q^2}}}$$



LEARNING EXAMPLE

Determine ω_0 , ω_{\max} when $R = 50\Omega$ and $R = 1\Omega$



Natural frequency depends only on L, C.
Resonant frequency depends on Q.

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

$$u_{\max} = \frac{\omega_{\max}}{\omega_0} = \sqrt{1 - \frac{1}{2Q^2}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(5 \times 10^{-2})(5 \times 10^{-6})}} = 2000 \text{ rad/s}$$

$$Q = \frac{2000 \times 0.050}{R}$$

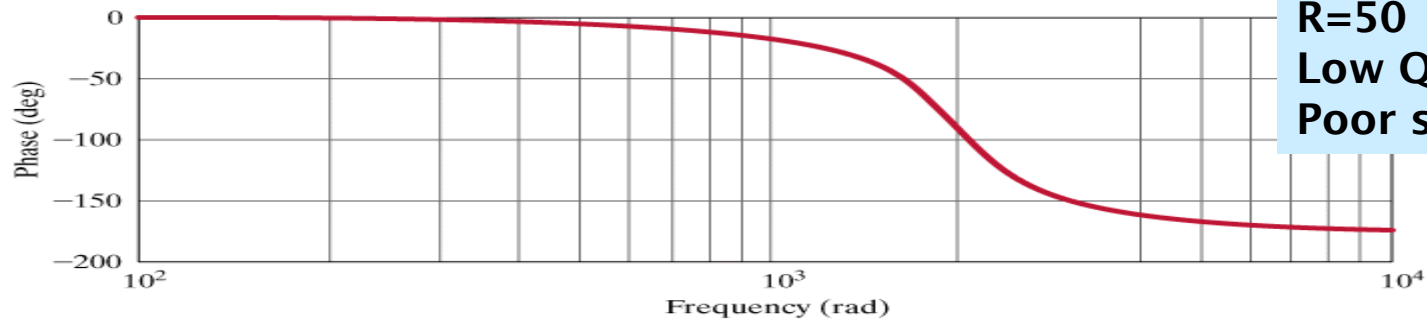
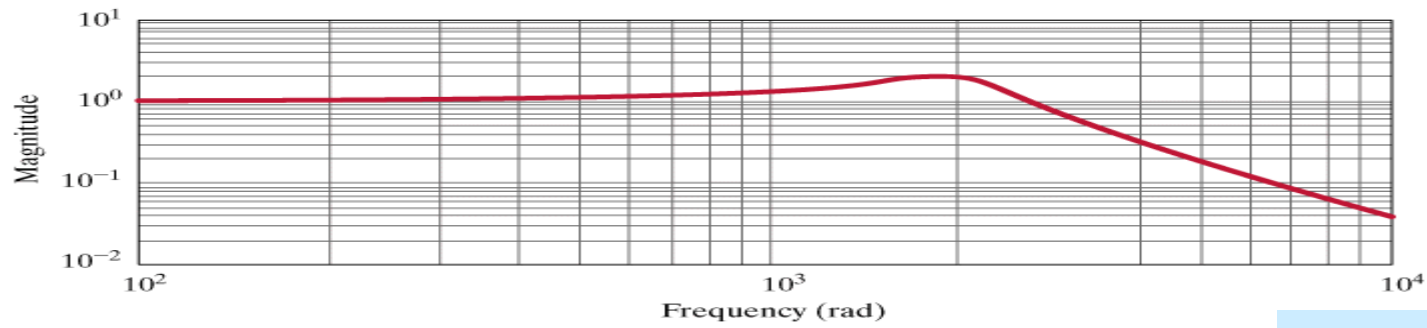
$$\omega_{\max} = 2000 \times \sqrt{1 - \frac{1}{2Q^2}}$$

R	Q	Wmax
50	2	1871
1	100	2000

Evaluated with EXCEL and rounded to zero decimals

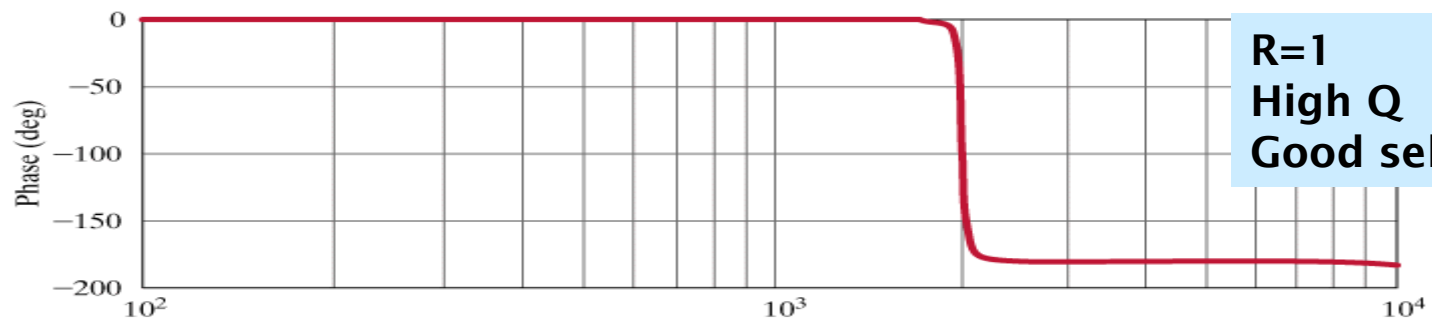
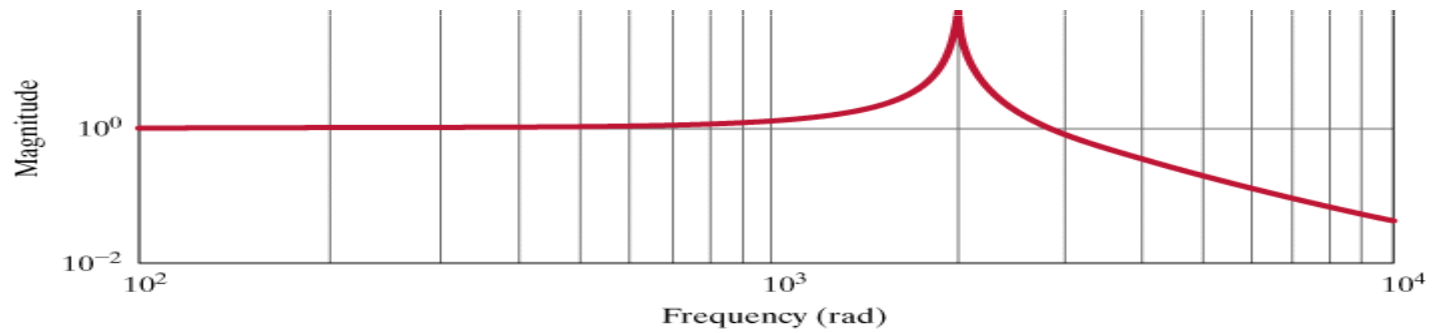
Using MATLAB one can display the frequency response



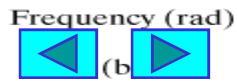


R=50
Low Q
Poor selectivity

(a)



R=1
High Q
Good selectivity



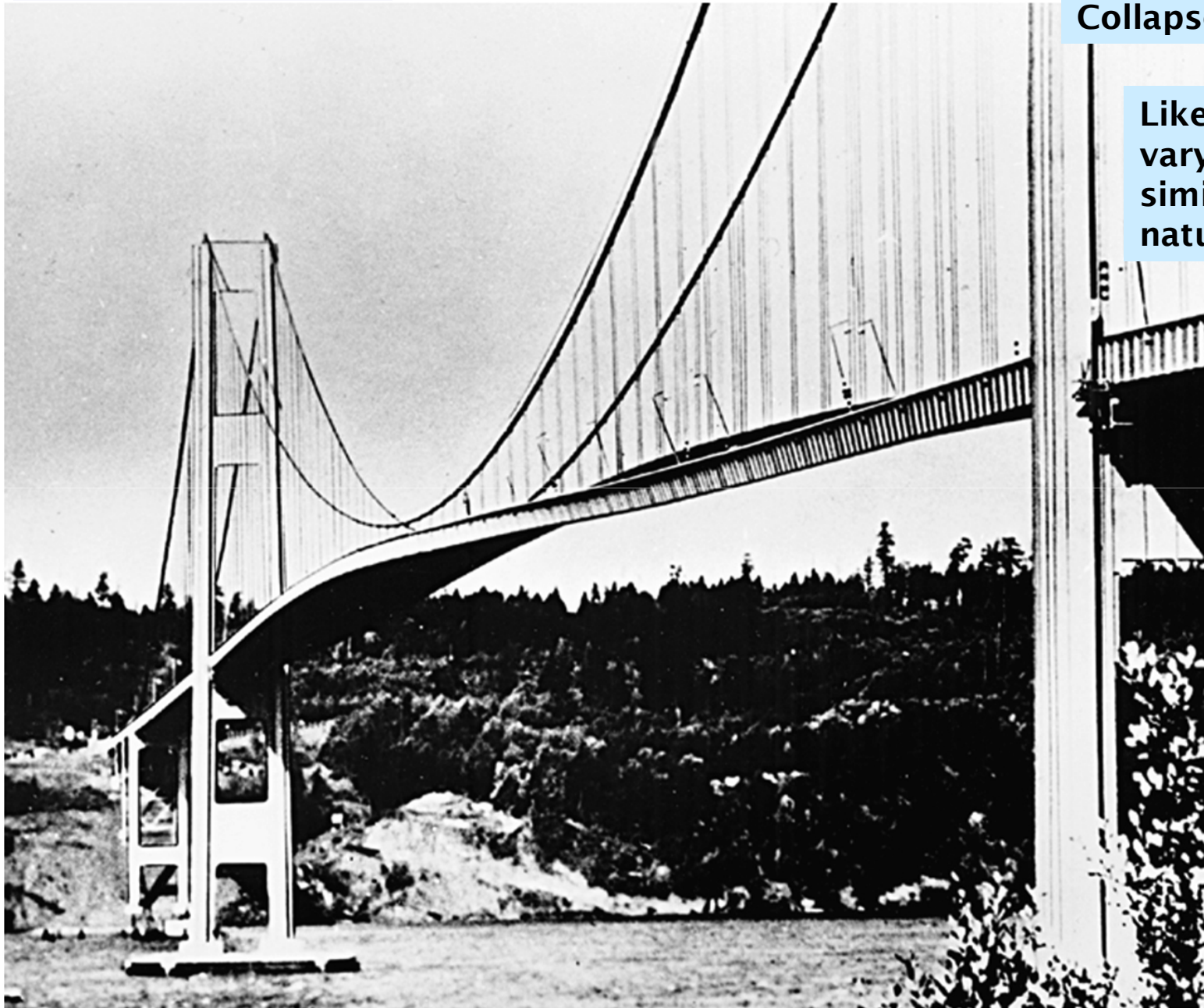
LEARNING EXAMPLE

The Tacoma Narrows Bridge

Opened: July 1, 1940
Collapsed: Nov 7, 1940

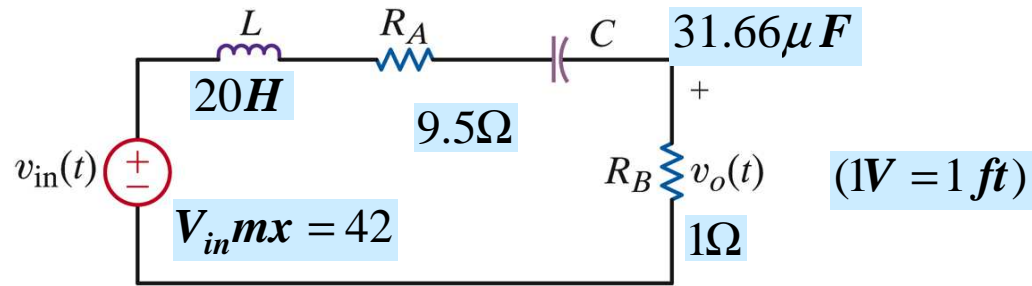
Likely cause: wind
varying at frequency
similar to bridge
natural frequency

$$\omega_0 = 2\pi \times 0.2$$



Tacoma Narrows Bridge Simulator

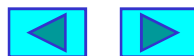
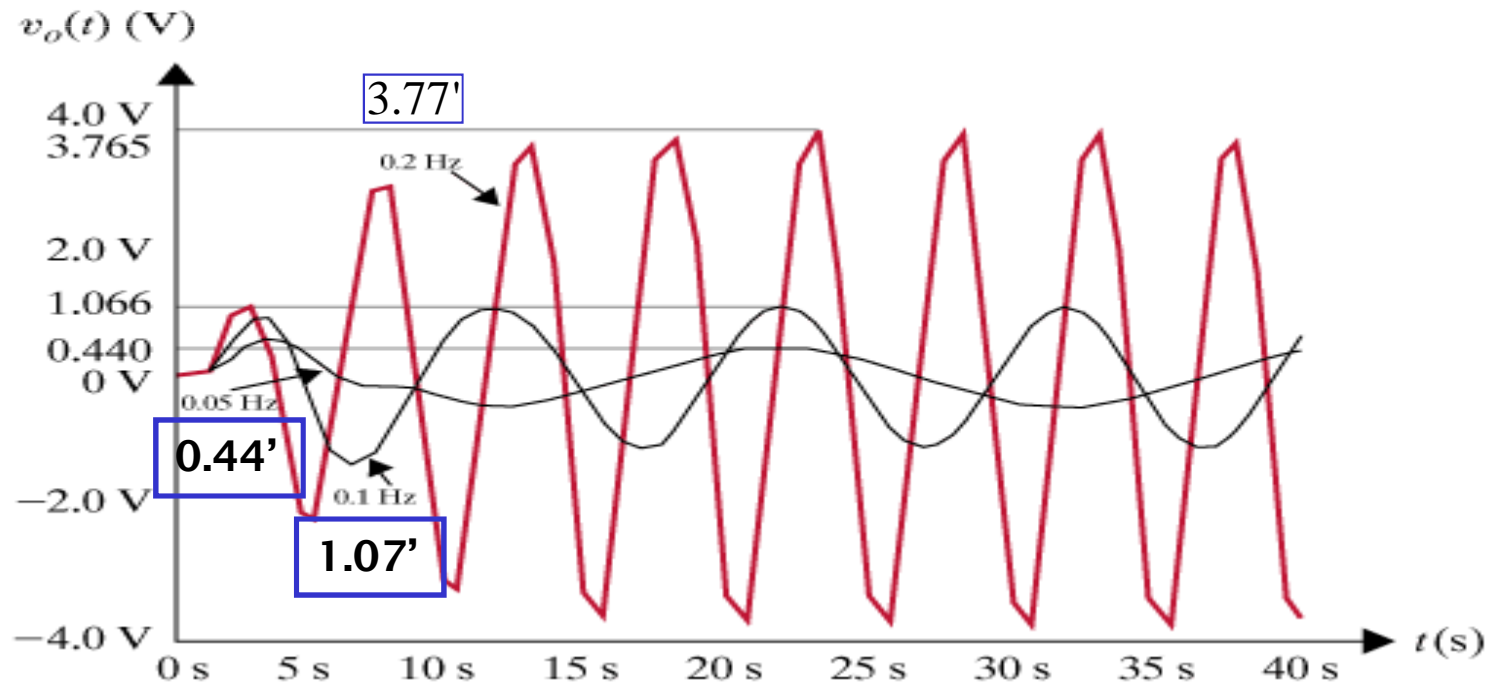
Assume a low $Q=2.39$



At failure a 42mph wind caused 4' deflection.

For the model at resonance

$$\frac{v_o}{v_{in}} = \frac{R_B}{R_A + R_B} \approx \frac{4}{42}$$



PARALLEL RLC RESONANT CIRCUITS

Impedance of series RLC

$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

$$|Z|^2 = R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2$$

Notice equivalences

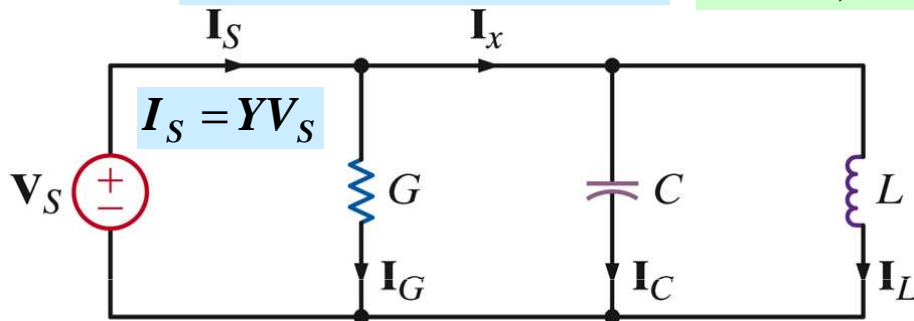
$$R \leftrightarrow G, L \leftrightarrow C, C \leftrightarrow L$$

$$Z \leftrightarrow Y, V \leftrightarrow I$$

Admittance of parallel RLC

$$Y(j\omega) = G + \frac{1}{j\omega L} + j\omega C$$

$$|Y|^2 = G^2 + \left(\omega C - \frac{1}{\omega L}\right)^2$$



Series RLC

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

Parallel RLC

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 LG}$$

At resonance

$$\omega_0 C = \frac{1}{\omega_0 L} \Rightarrow Y = G$$

$$I_G = I_S$$

$$I_C = -I_L$$

$$|I_C| = \frac{\omega_0 C}{G} |I_S| = Q |I_S|$$

$$|I_L| = \frac{1}{\omega_0 LG} |I_S|$$

$$I_G = G V_S = \frac{G}{Y} I_S$$

$$I_C = j\omega C V_S = \frac{j\omega C}{Y} I_S$$

$$I_L = \frac{1}{j\omega L} V_S = \frac{1/j\omega L}{Y} I_S$$

Series RLC

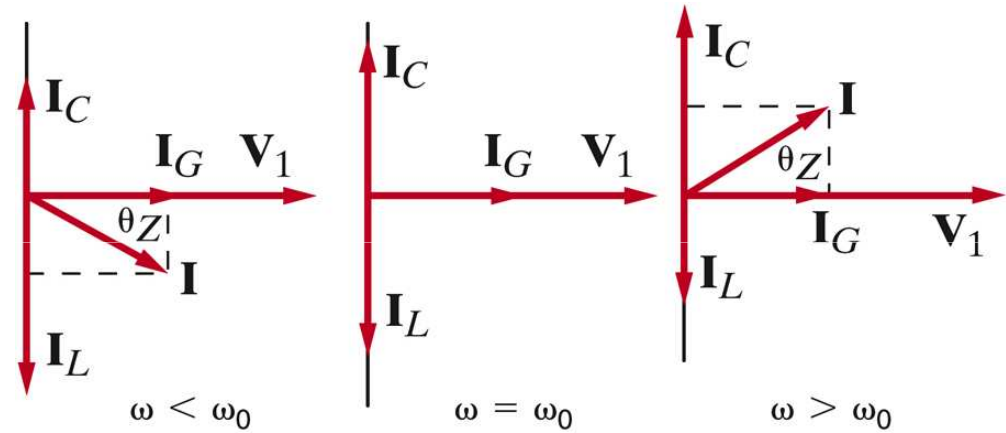
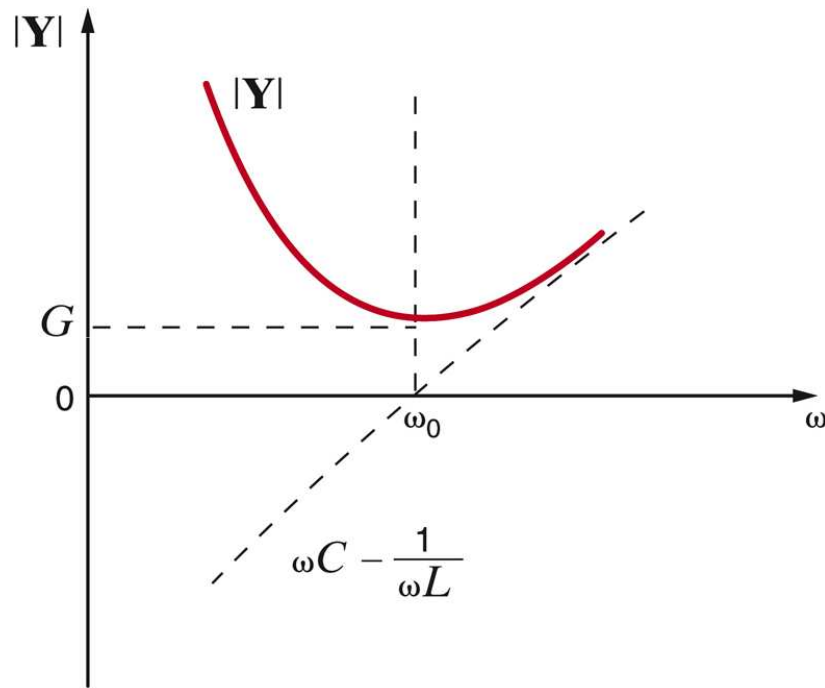
$$BW = \frac{\omega_0}{Q}$$

Parallel RLC

$$BW = \frac{\omega_0}{Q}$$

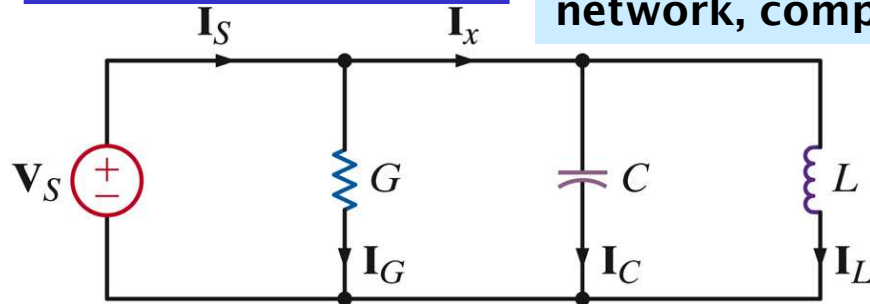


VARIATION OF IMPEDANCE AND PHASOR DIAGRAM - PARALLEL CIRCUIT



LEARNING EXAMPLE

If the source operates at the resonant frequency of the network, compute all the branch currents



$$I_G = 0.01 \times 120 \angle 0^\circ = 1.2 \angle 0^\circ (\text{A}) = I_S$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.120 \times (6 \times 10^{-4})}} = 117.85 \text{ rad/s}$$

$$V_S = 120 \angle 0^\circ, \quad G = 0.01 \text{ S}$$

$$C = 600 \mu\text{F}, \quad L = 120 \text{ mH}$$

$$I_C = (1 \angle 90^\circ) \times (117.85) \times (600 \times 10^{-6}) \times 120 \angle 0^\circ = 8.49 \angle 90^\circ (\text{A})$$

$$I_L = 8.49 \angle -90^\circ (\text{A})$$

At resonance

$$\omega_0 C = \frac{1}{\omega_0 L} \Rightarrow Y = G$$

$$I_G = I_S$$

$$I_C = -I_L$$

$$|I_C| = \frac{\omega_0 C}{G} |I_S| = Q |I_S|$$

$$|I_L| = \frac{1}{\omega_0 L G} |I_S|$$

$$I_G = G V_S = \frac{G}{Y} I_S$$

$$I_C = j\omega C V_S = \frac{j\omega C}{Y} I_S$$

$$I_L = \frac{1}{j\omega L} V_S = \frac{1}{j\omega L Y} I_S$$

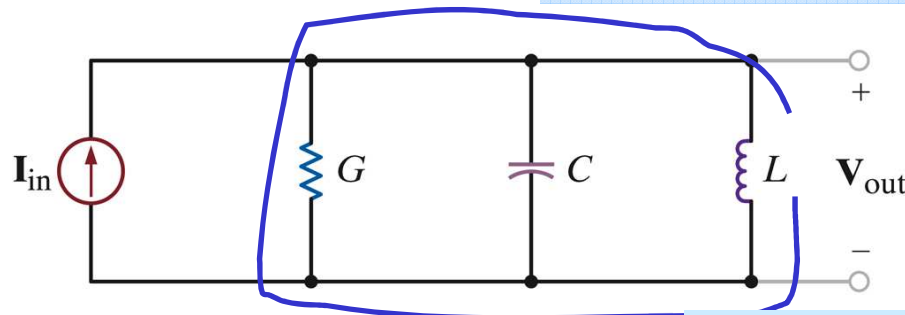
$$I_x = \underline{\hspace{2cm}}$$



LEARNING EXAMPLE

Derive expressions for the resonant frequency, half power frequencies, bandwidth and quality factor for the transfer characteristic

$$H = \frac{V_{out}}{I_{in}}$$



$$V_{out} = \frac{I_{in}}{Y_T} \Rightarrow H = \frac{V_{out}}{I_{in}} = \frac{1}{Y_T}$$

$$Y_T = G + j\omega C + \frac{1}{j\omega L}$$

$$|H| = \frac{1}{\left| G + j\omega C + \frac{1}{j\omega L} \right|} = \frac{1}{\sqrt{G^2 + \left(\omega C - \frac{1}{\omega L} \right)^2}}$$

Resonant frequency: $\omega_0 = \frac{1}{\sqrt{LC}}$ $|H_{max}| = \frac{1}{G} = R$

Half power frequencies $\Rightarrow |H(j\omega_h)|^2 = 0.5 |H|_{max}^2$

$$G^2 + \left(\omega_h C - \frac{1}{\omega_h L} \right)^2 = 2G^2 \Rightarrow \omega_h C - \frac{1}{\omega_h L} = \pm G$$

$$\omega_h = \mp \frac{G}{2C} + \sqrt{\left(\frac{G}{2C} \right)^2 + \frac{1}{LC}}$$

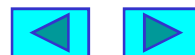
$$BW = \omega_{HI} - \omega_{LO} = \frac{G}{C}$$

$$Q = \frac{\omega_0}{BW} = \frac{1}{G} \sqrt{\frac{C}{L}} = R \sqrt{\frac{C}{L}}$$

Replace and show

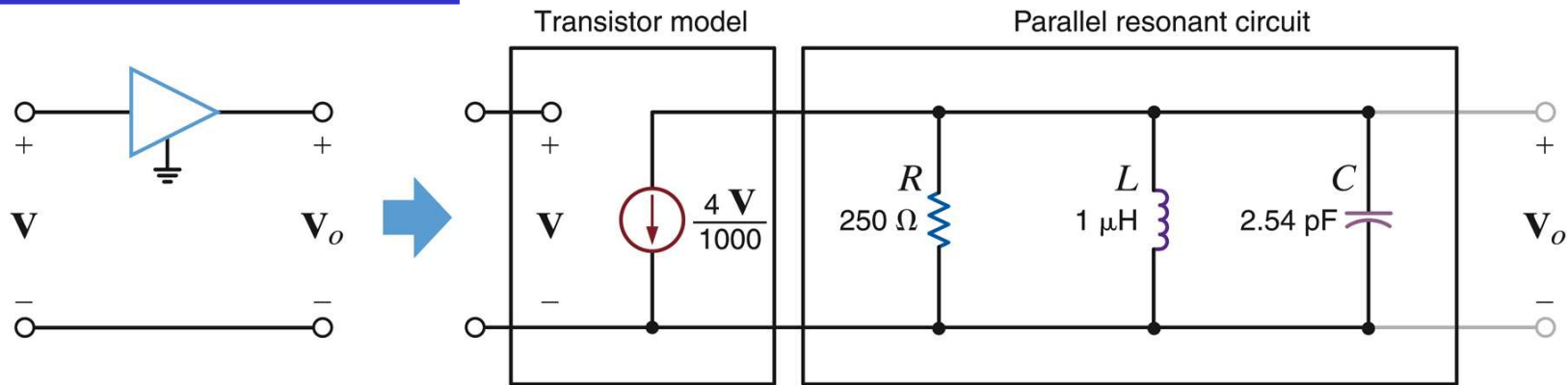
$$Q = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 L G}$$

$$\omega_{LO} = \omega_0 \left[-\frac{1}{2Q} + \sqrt{\left(\frac{1}{2Q} \right)^2 + 1} \right]$$



LEARNING EXAMPLE

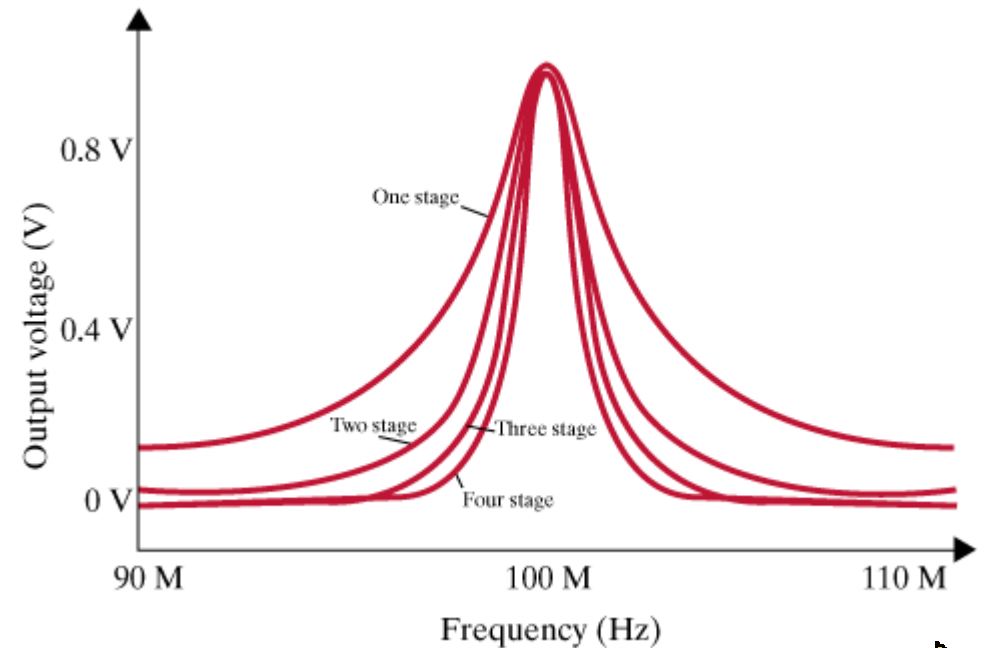
Increasing selectivity by cascading low Q circuits



$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(10^{-6} \text{ H})(2.54 \times 10^{-12} \text{ F})}} = 6.275 \times 10^8 \text{ rad/s} = 99.9 \text{ MHz}$$

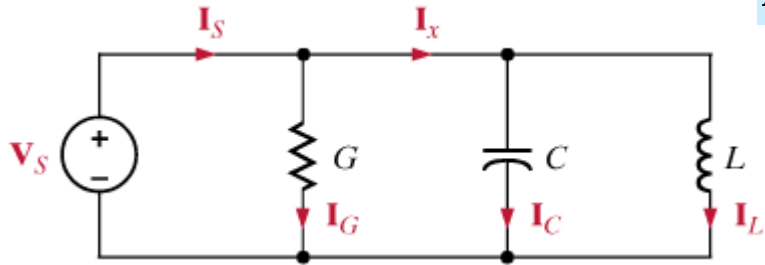
$$Q = \frac{\omega_0}{BW} = \frac{1}{G} \sqrt{\frac{C}{L}} = R \sqrt{\frac{C}{L}}$$

$$= 250 \times \sqrt{\frac{2.54 \times 10^{-12}}{10^{-6}}} = 0.398$$



LEARNING EXTENSION**Determine the resonant frequency, Q factor and bandwidth**

$$R = 2k\Omega, L = 20mH, C = 150\mu F$$

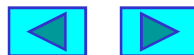
**Parallel RLC**

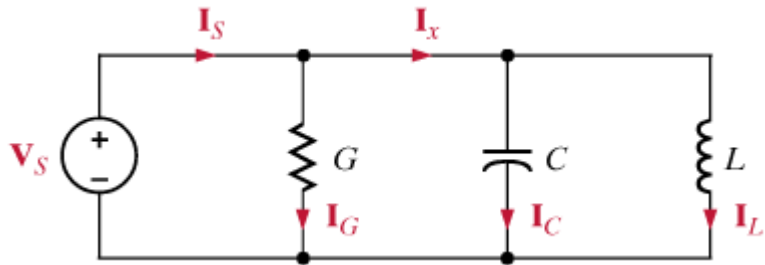
$$\omega_0 = \frac{1}{\sqrt{LC}} \quad Q = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 LG} \quad BW = \frac{\omega_0}{Q}$$

$$\omega_0 = \frac{1}{\sqrt{(20 \times 10^{-3})(150 \times 10^{-6})}} = 577 \text{ rad/s}$$

$$Q = \frac{577 \times 150 \times 10^{-6}}{(1/2000)} = 173$$

$$BW = \frac{577}{173} = 3.33 \text{ rad/s}$$



LEARNING EXTENSIONDetermine L , C , ω_0 

$$R = 6k\Omega, BW = 1000rad/s, Q = 120$$

Parallel RLC

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

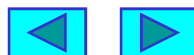
$$Q = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 LG} \quad BW = \frac{\omega_0}{Q}$$

$$\omega_0 = Q \times BW = 120 \times 1000 = 1.2 \times 10^5 rad/s$$

$$C = \frac{Q}{R \omega_0} = \frac{120}{6000 \times 1.2 \times 10^5} = 0.167 \mu F$$

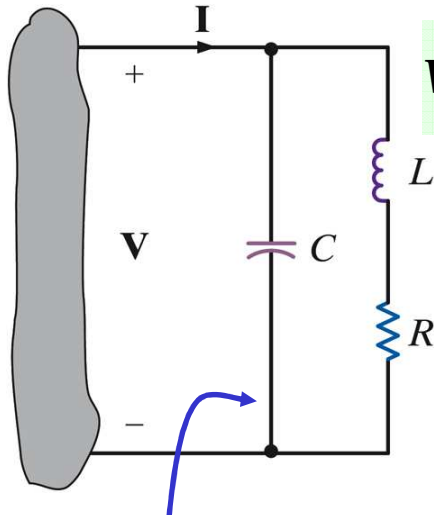
$$L = \frac{R}{Q \omega_0} = \frac{6000}{120 \times 1.2 \times 10^5} = 417 \mu H$$

Can be used to verify computations



PRACTICAL RESONANT CIRCUIT

The resistance of the inductor coils cannot be neglected



$V = ZI = \frac{I}{Y}$. At resonance the voltage and impedance are maxima

$$Z_{MAX} = \frac{R^2 + (\omega_R L)^2}{R} = R \left(1 + \left(\frac{\omega_R L}{R} \right)^2 \right) = R \left(1 + \left(\frac{\omega_R}{\omega_0} \right)^2 \left(\frac{\omega_0 L}{R} \right)^2 \right)$$

$$Z_{MAX} = RQ_0^2$$

$$Y(j\omega) = j\omega C + \frac{1}{R + j\omega L} \times \frac{R - j\omega L}{R - j\omega L}$$

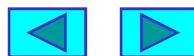
$$Y(j\omega) = j\omega C + \frac{R - j\omega L}{R^2 + (\omega L)^2}$$

$$Y(j\omega) = \frac{R}{R^2 + (\omega L)^2} + j \left(C - \frac{\omega L}{R^2 + (\omega L)^2} \right)$$

$$Y \text{ real} \Rightarrow C - \frac{\omega L}{R^2 + (\omega L)^2} = 0 \Rightarrow \omega_R = \sqrt{\frac{1}{LC} - \left(\frac{R}{L} \right)^2}$$

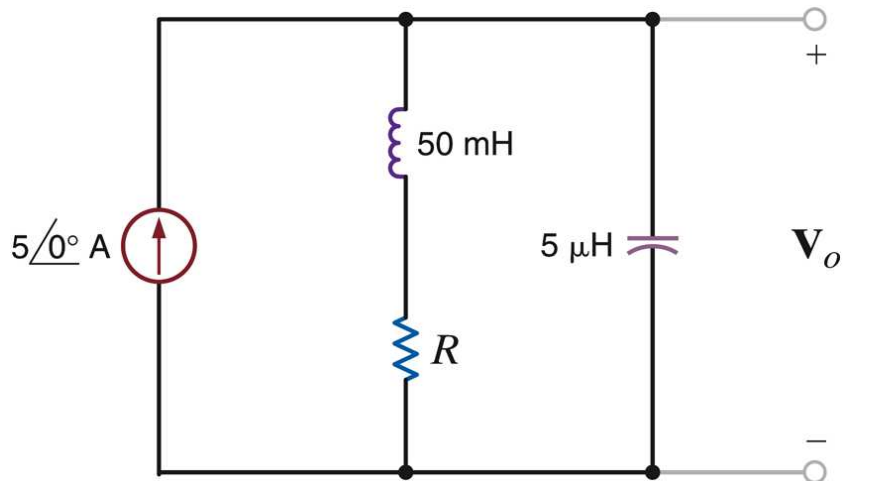
$$\omega_0 = \frac{1}{\sqrt{LC}}, Q_0 = \frac{\omega_0 L}{R} \Rightarrow \omega_R = \omega_0 \sqrt{1 - \frac{1}{Q_0^2}}$$

How do you define a quality factor for this circuit?



LEARNING EXAMPLE

Determine both ω_0 , ω_R for $R = 50\Omega, 5\Omega$

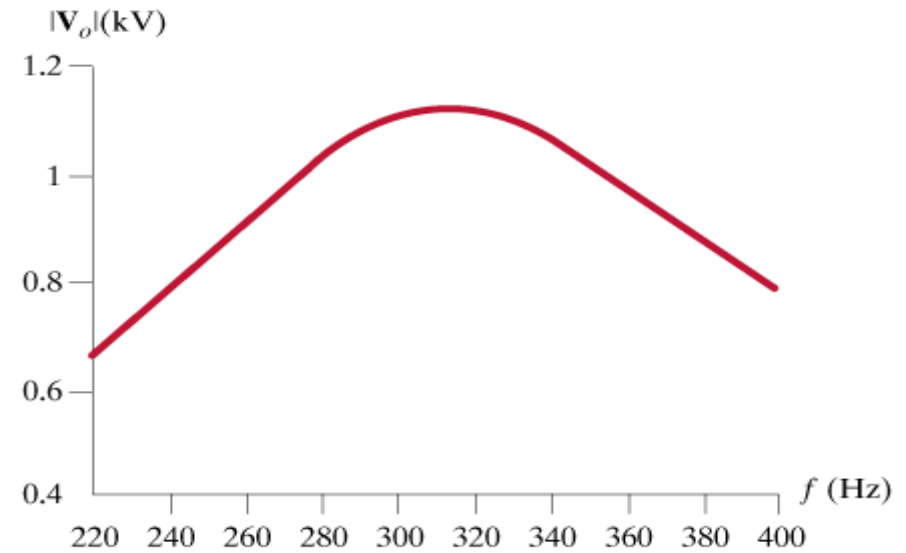


$$\omega_0 = \frac{1}{\sqrt{LC}}, Q_0 = \frac{\omega_0 L}{R} \Rightarrow \omega_R = \omega_0 \sqrt{1 - \frac{1}{Q_0^2}}$$

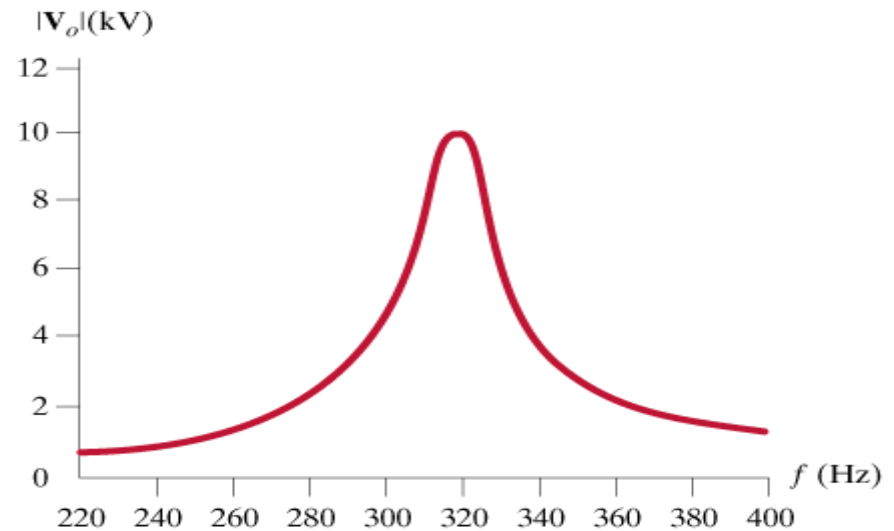
$$\omega_0 = \frac{1}{\sqrt{(50 \times 10^{-3} \text{ H})(5 \times 10^{-6} \text{ F})}} = 2000 \text{ rad/s}$$

$$Q_0 = \frac{2000 \times 0.050}{R}, \omega_R = 2000 \sqrt{1 - \frac{1}{Q_0^2}}$$

R	Q0	Wr(rad/s)	f(Hz)
50	2	1732	275.7
5	20	1997	317.8



(a) $R = 50 \Omega$



(b) $R = 5 \Omega$



RESONANCE IN A MORE GENERAL VIEW

$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

$$|Z|^2 = R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2$$

$$Y(j\omega) = G + \frac{1}{j\omega L} + j\omega C$$

$$|Y|^2 = G^2 + \left(\omega C - \frac{1}{\omega L}\right)^2$$

For series connection the impedance reaches maximum at resonance. For parallel connection the impedance reaches maximum

$$Y_s = \frac{j\omega C}{(j\omega)^2 LC + j\omega CR + 1} \quad Z_p = \frac{j\omega L}{(j\omega)^2 LC + j\omega LG + 1}$$

In Bode plots the quadratic term was written as

$$(j\omega\tau)^2 + 2\zeta j\omega\tau + 1$$

$$\tau = \sqrt{LC} = \frac{1}{\omega_0}$$

series

$$2\zeta\tau = CR \Rightarrow 2\zeta = \omega_0 CR = \frac{1}{Q}$$

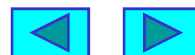
parallel

$$2\zeta\tau = LG \Rightarrow 2\zeta = \omega_0 LG = \frac{1}{Q}$$

A high Q circuit is highly under damped

$$Q = \frac{1}{2\zeta}$$

Resonance



SCALING

Scaling techniques are used to change an idealized network into a more realistic one or to adjust the values of the components

Magnitude or impedance scaling

$$R' \rightarrow K_M R$$

$$L' \rightarrow K_M L$$

$$C' \rightarrow \frac{C}{K_M}$$

$$LC = L'C' \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{L'C'}}$$

$$Q = \frac{\omega_0 L}{R} = \frac{\omega_0 L'}{R'}$$

Magnitude scaling does not change the frequency characteristics nor the quality of the network.

Frequency or time scaling

$$\omega' \rightarrow K_F \omega$$

Impedance of each component is unchanged

$$\omega' L' = \omega L, \quad \frac{1}{\omega' C'} = \frac{1}{\omega C}$$

$$R' \rightarrow R$$

$$L' \rightarrow \frac{L}{K_F}$$

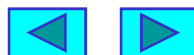
$$C' \rightarrow \frac{C}{K_F}$$

$$\omega'_0 = K_F \omega_0$$

$$Q' = \frac{\omega'_0 L'}{R'} = Q$$

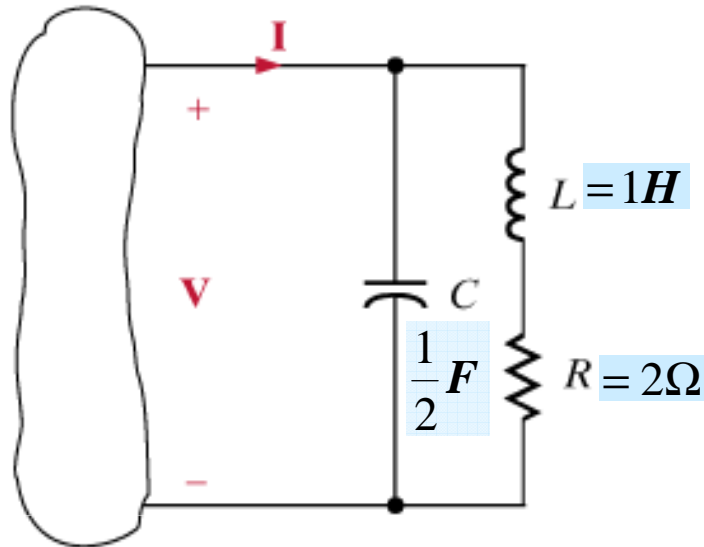
Constant Q networks

$$BW' = \frac{\omega'_0}{Q'} = K_F (BW)$$



LEARNING EXAMPLE

Determine the value of the elements and the characteristics of the network if the circuit is magnitude scaled by 100 and frequency scaled by 1,000,000



$$\omega_0 = \sqrt{2} \text{ rad/s}, \quad Q = \frac{\sqrt{2}}{2}, \quad BW = 2$$

$$R' \rightarrow R$$

$$L' \rightarrow \frac{L}{K_F}$$

$$C' \rightarrow \frac{C}{K_F}$$

$$R'' = 200 \Omega$$

$$L'' = 100 \text{ mH}$$

$$C'' = \frac{1}{200} \mu F$$

Magnitude or impedance scaling

$$R' \rightarrow K_M R$$

$$L' \rightarrow K_M L$$

$$C' \rightarrow \frac{C}{K_M}$$

$$\omega_0' = K_F \omega_0 \quad \omega_0'' = 1.414 \times 10^6 \text{ rad/s}$$

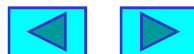
$$BW' = \frac{\omega_0'}{Q'} = K_F (BW)$$

$$R' = 200 \Omega$$

$$L' = 100 H$$

$$C' = \frac{1}{200} F$$

Q, ω_0 are unchanged



LEARNING EXTENSION

An RLC network with $R = 10\Omega$, $L = 1H$, $C = 2F$ is magnitude scaled by 100 and frequency scaled by 10,000. Determine the resulting circuit elements

Magnitude or impedance scaling

$$R' \rightarrow K_M R$$

$$R' = 1000\Omega$$

$$L' \rightarrow K_M L$$

$$L' = 100H$$

$$C' \rightarrow \frac{C}{K_M}$$

$$C' = 0.02F$$

Frequency scaling

$$R' \rightarrow R$$

$$R'' = 1k\Omega$$

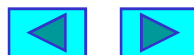
$$L' \rightarrow \frac{L}{K_F}$$

$$L'' = 0.01H$$

$$C' \rightarrow \frac{C}{K_F}$$

$$C'' = 2\mu F$$

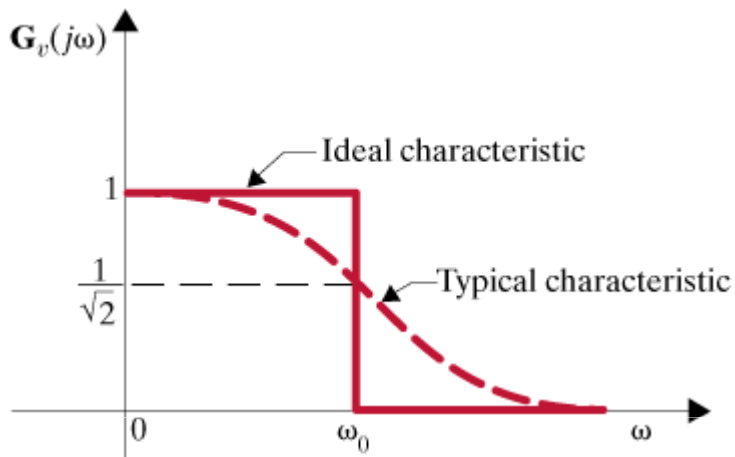
Scaling



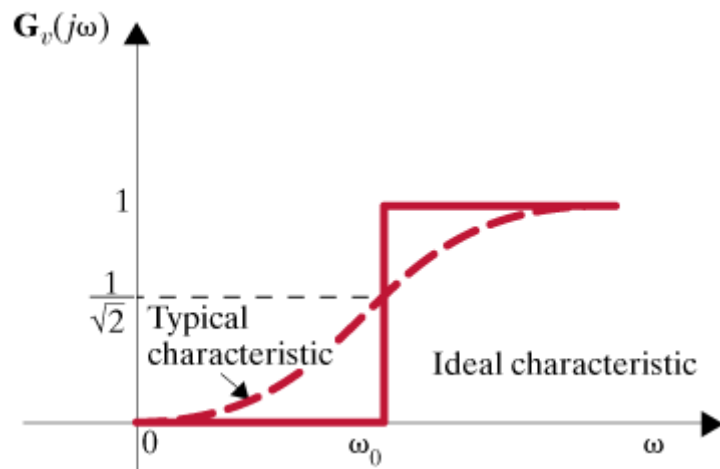
FILTER NETWORKS

Networks designed to have frequency selective behavior

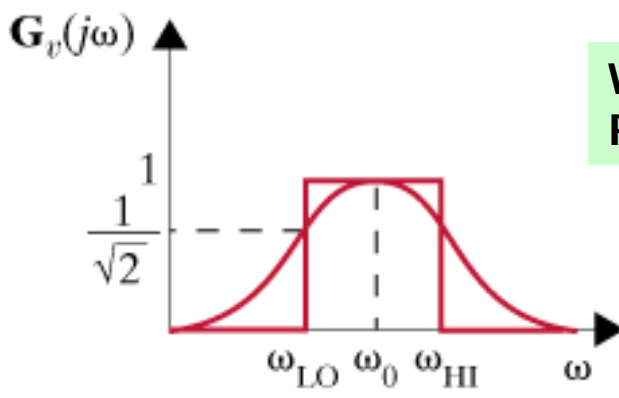
COMMON FILTERS



Low-pass filter

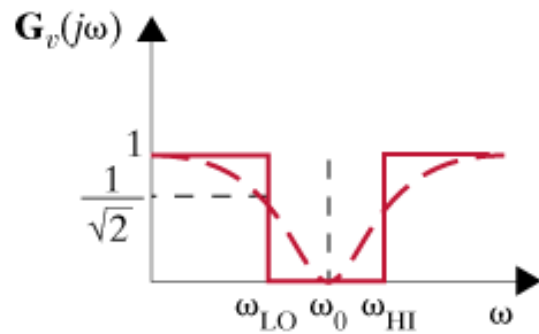


High-pass filter



Band-pass filter

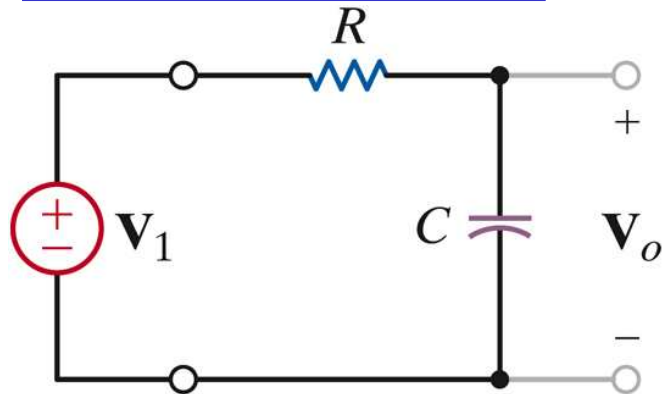
We focus first on PASSIVE filters



Band-reject filter



Simple low-pass filter



$$G_v = \frac{V_0}{V_1} = \frac{1}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

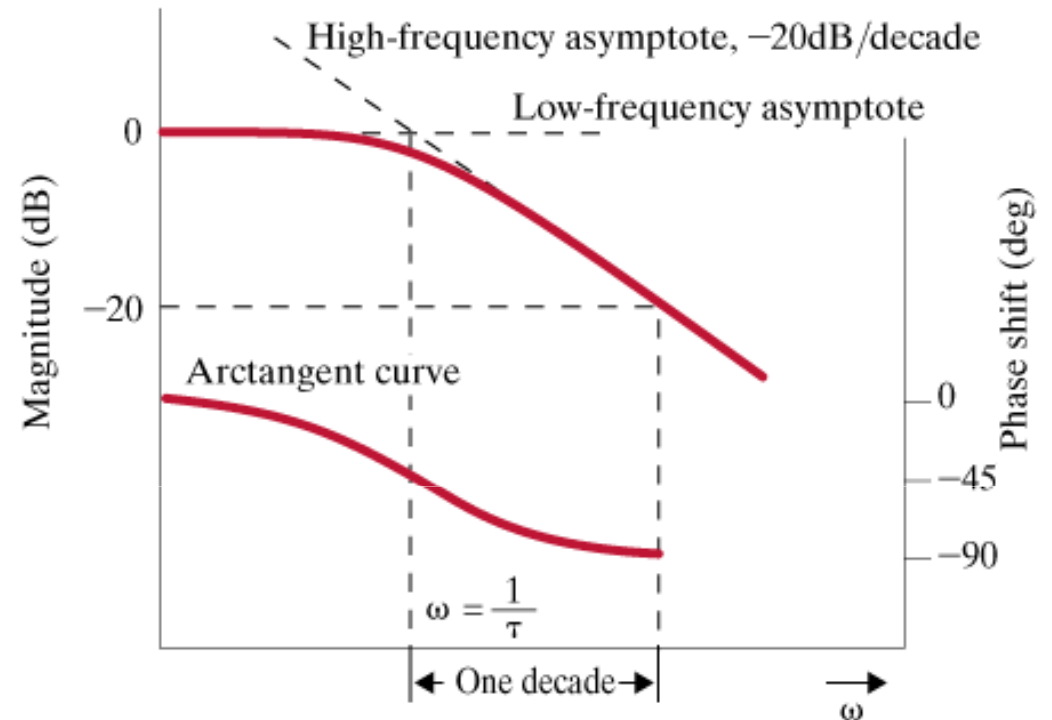
$$G_v = \frac{1}{1 + j\omega\tau}; \quad \tau = RC$$

$$M(\omega) = |G_v| = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

$$\angle G_v = \phi(\omega) = -\tan^{-1} \omega\tau$$

$$M_{\max} = 1, \quad M\left(\omega = \frac{1}{\tau}\right) = \frac{1}{\sqrt{2}}$$

$$\omega = \frac{1}{\tau} = \text{half power frequency}$$

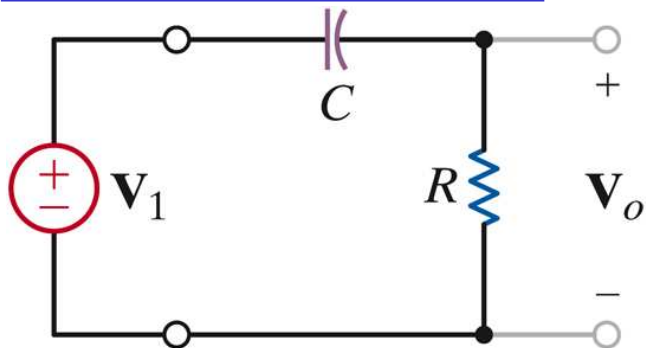


(c)

$$BW = \frac{1}{\tau}$$



Simple high-pass filter



$$G_v = \frac{V_0}{V_1} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega CR}{1 + j\omega CR}$$

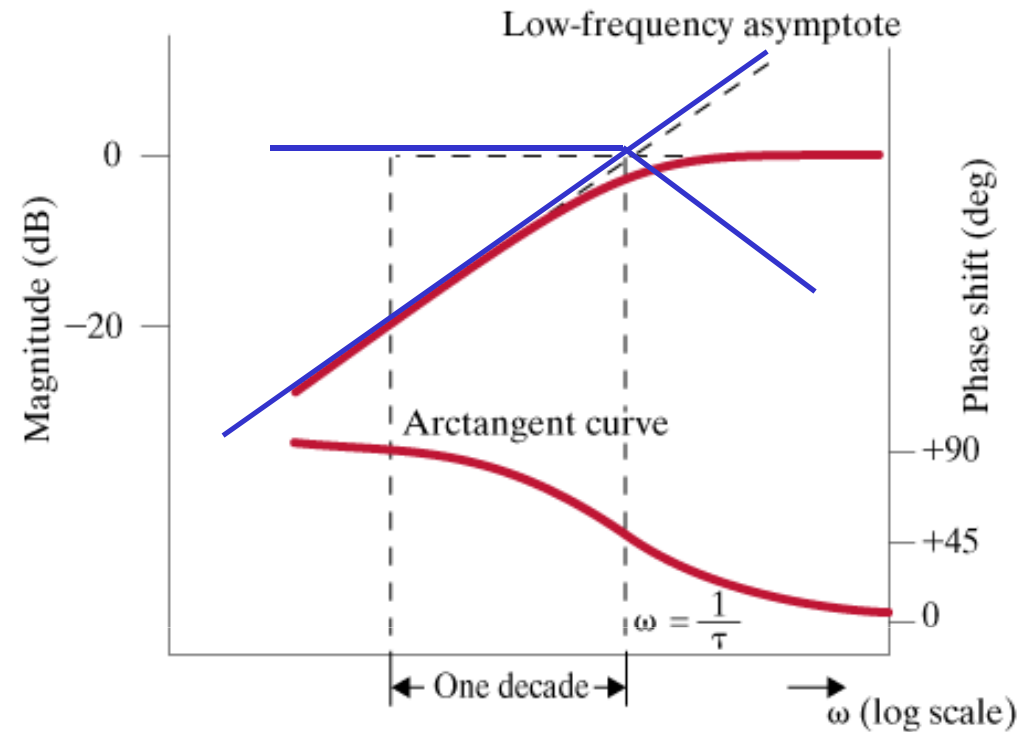
$$G_v = \frac{j\omega\tau}{1 + j\omega\tau}; \quad \tau = RC$$

$$M(\omega) = |G_v| = \frac{\omega\tau}{\sqrt{1 + (\omega\tau)^2}}$$

$$\angle G_v = \phi(\omega) = \frac{\pi}{2} - \tan^{-1} \omega\tau$$

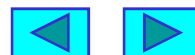
$$M_{\max} = 1, \quad M\left(\omega = \frac{1}{\tau}\right) = \frac{1}{\sqrt{2}}$$

$$\omega = \frac{1}{\tau} = \text{half power frequency}$$

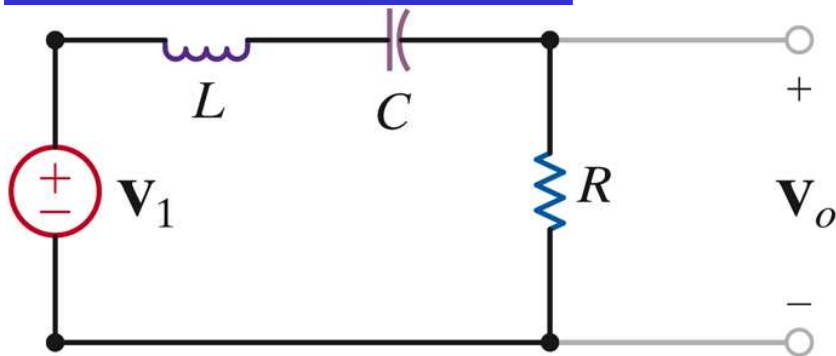


(c)

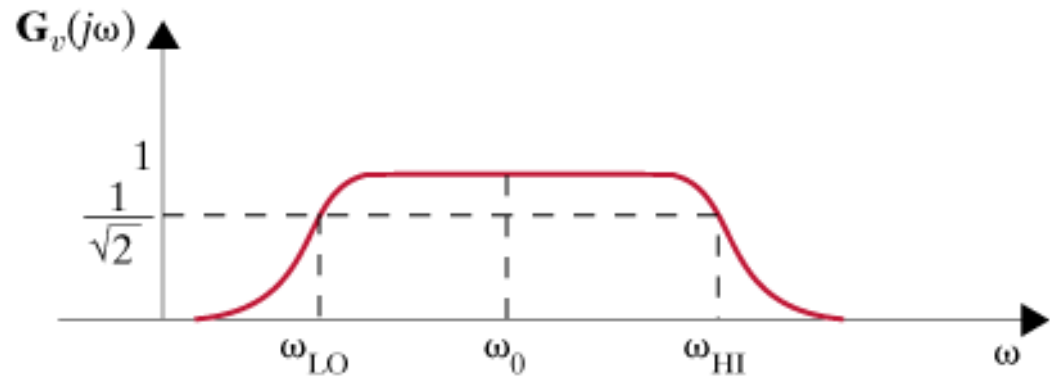
$$\omega_{LO} = \frac{1}{\tau}$$



Simple band-pass filter



Band-pass



(e)

$$G_v = \frac{V_0}{V_1} = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$M(\omega) = \frac{\omega RC}{\sqrt{(\omega RC)^2 + (\omega^2 LC - 1)^2}}$$

$$M\left(\omega = \frac{1}{\sqrt{LC}}\right) = 1 \quad M(\omega = 0) = M(\omega = \infty) = 0$$

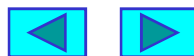
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$M(\omega_{LO}) = \frac{1}{\sqrt{2}} = M(\omega_{HI})$$

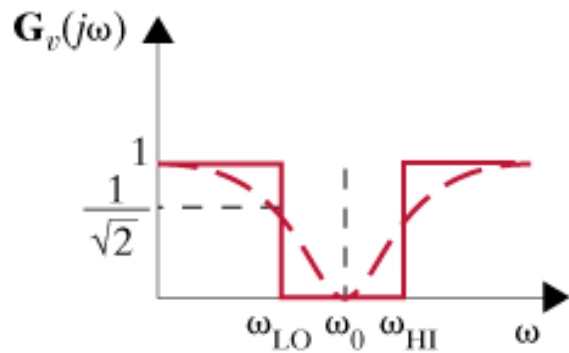
$$\omega_{LO} = \frac{-(R/L) + \sqrt{(R/L)^2 + 4\omega_0^2}}{2}$$

$$\omega_{HI} = \frac{(R/L) + \sqrt{(R/L)^2 + 4\omega_0^2}}{2}$$

$$BW = \omega_{HI} - \omega_{LO} = \frac{R}{L}$$



Simple band-reject filter

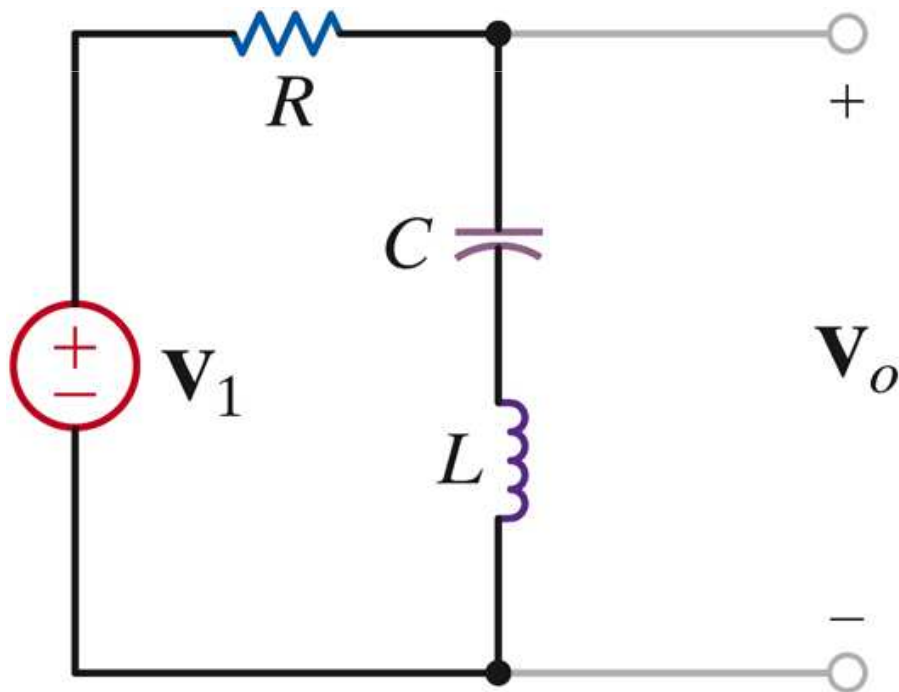


(b)

$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow j\left(\omega_0 L - \frac{1}{\omega_0 C}\right) = 0$$

at $\omega = 0$ the capacitor acts as open circuit $\Rightarrow V_0 = V_1$

at $\omega = \infty$ the inductor acts as open circuit $\Rightarrow V_0 = V_1$

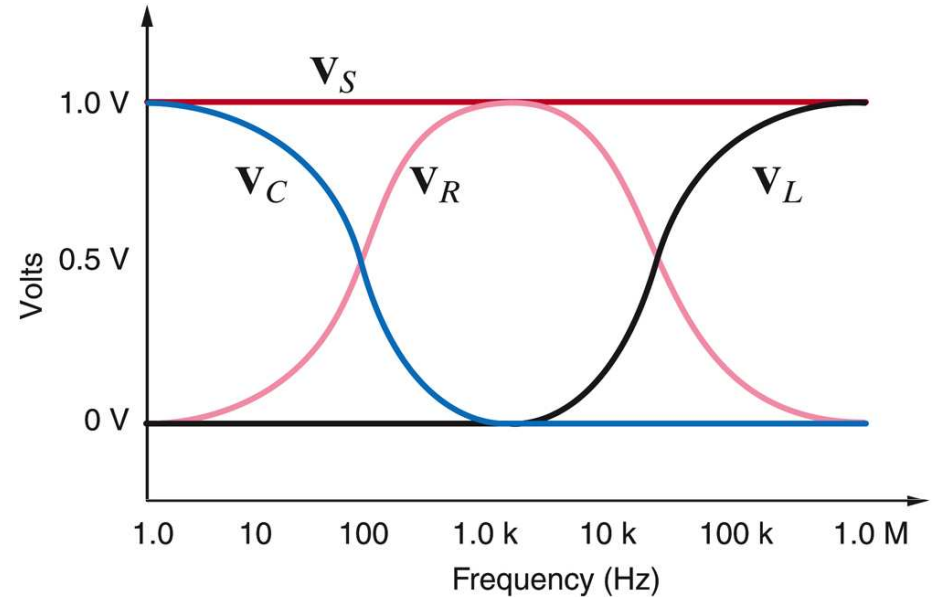
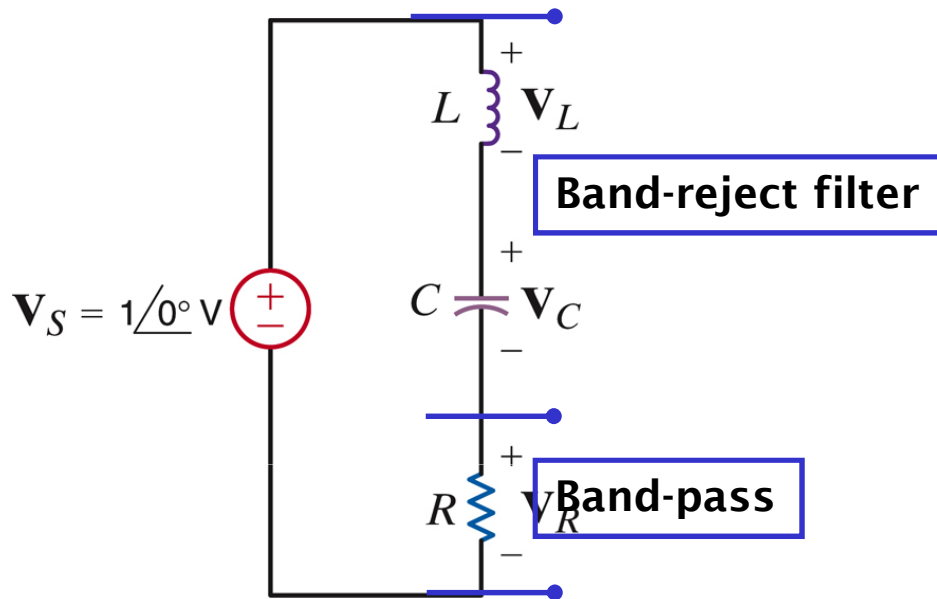


ω_{LO}, ω_{HI} are determined as in the band-pass filter



LEARNING EXAMPLE

Depending on where the output is taken, this circuit can produce low-pass, high-pass or band-pass or band-reject filters



Bode plot for $R = 10\Omega$, $L = 159\mu\text{H}$, $C = 159\mu\text{F}$

$$\frac{V_L}{V_S} = \frac{j\omega L}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$\frac{V_L}{V_S}(\omega = 0) = 0, \frac{V_L}{V_S}(\omega = \infty) = 1$$

High-pass

$$\frac{V_C}{V_S} = \frac{1}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

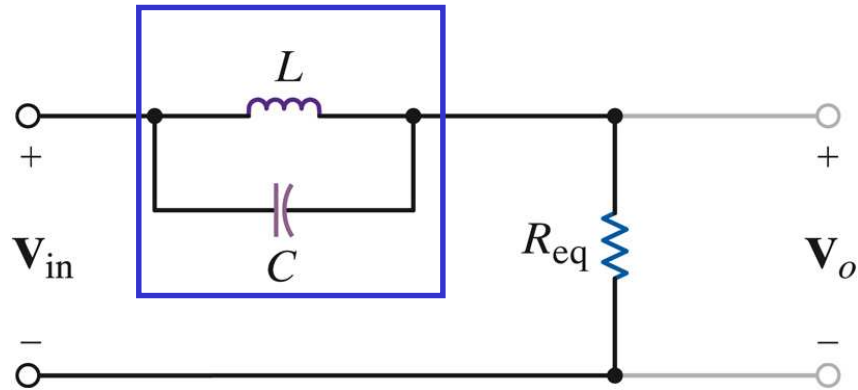
$$\frac{V_C}{V_S}(\omega = 0) = 1, \frac{V_C}{V_S}(\omega = \infty) = 0$$

Low-pass



LEARNING EXAMPLE

A simple notch filter to eliminate 60Hz interference



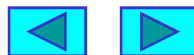
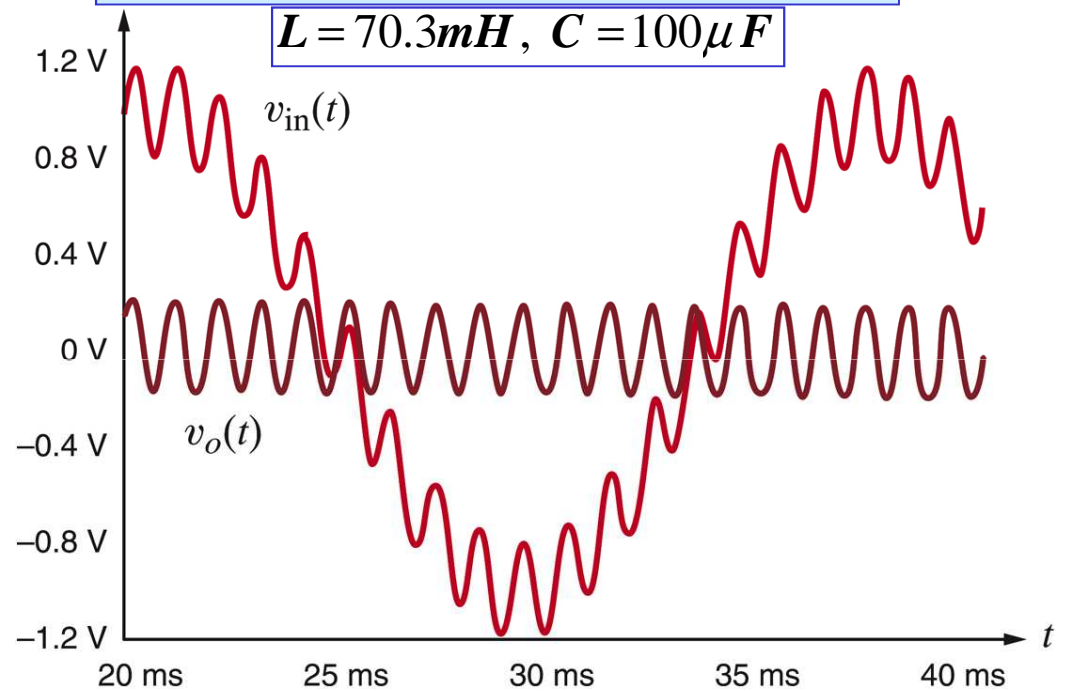
$$Z_R = \frac{j\omega L \frac{1}{j\omega C}}{j\omega L + \frac{1}{j\omega C}} = \frac{L/C}{j(\omega L - \frac{1}{\omega C})}$$

$$V_o = \frac{R_{eq}}{R_{eq} + Z_R} V_{in}$$

$$Z_R\left(\omega = \frac{1}{\sqrt{LC}}\right) = \infty \quad \therefore V_o\left(\omega = \frac{1}{\sqrt{LC}}\right) = 0$$

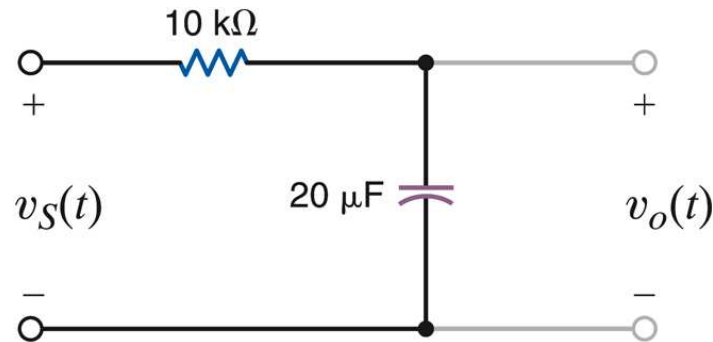
$$v_{in}(t) = \sin(2\pi \times 60t) + 0.2 \sin(2\pi \times 1000t)$$

$$L = 70.3mH, C = 100\mu F$$



LEARNING EXTENSION

Sketch the magnitude characteristic of the Bode plot for $G_v(j\omega)$



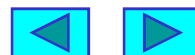
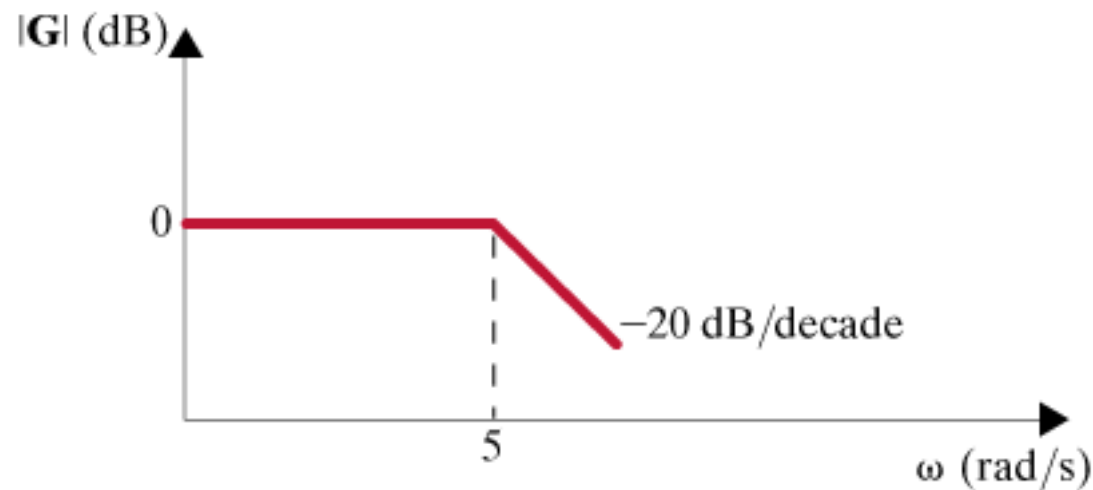
$$\tau = RC = (10 \times 10^3 \Omega)(20 \times 10^{-6} \text{ F}) = 0.2 \text{ rad/s}$$

$$G_v(j\omega) = \frac{1}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

Break/corner frequency : 5 rad/s

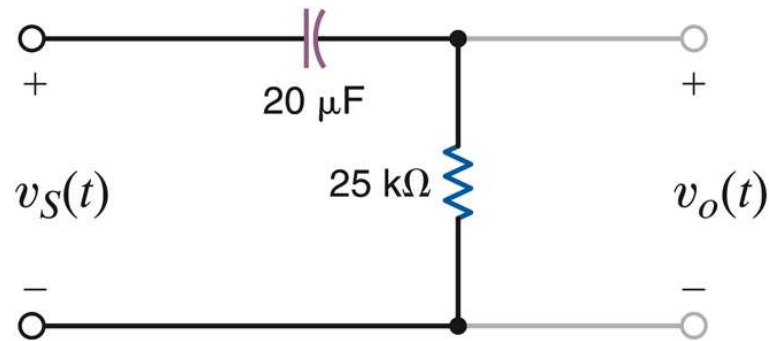
low frequency asymptote of 0 dB/dec

High frequency asymptote of -20 dB/dec



LEARNING EXTENSION

Sketch the magnitude characteristic of the Bode plot for $G_v(j\omega)$



$$\tau = RC = (25 \times 10^3 \Omega)(20 \times 10^{-6} \text{F}) = 0.5 \text{rad/s}$$

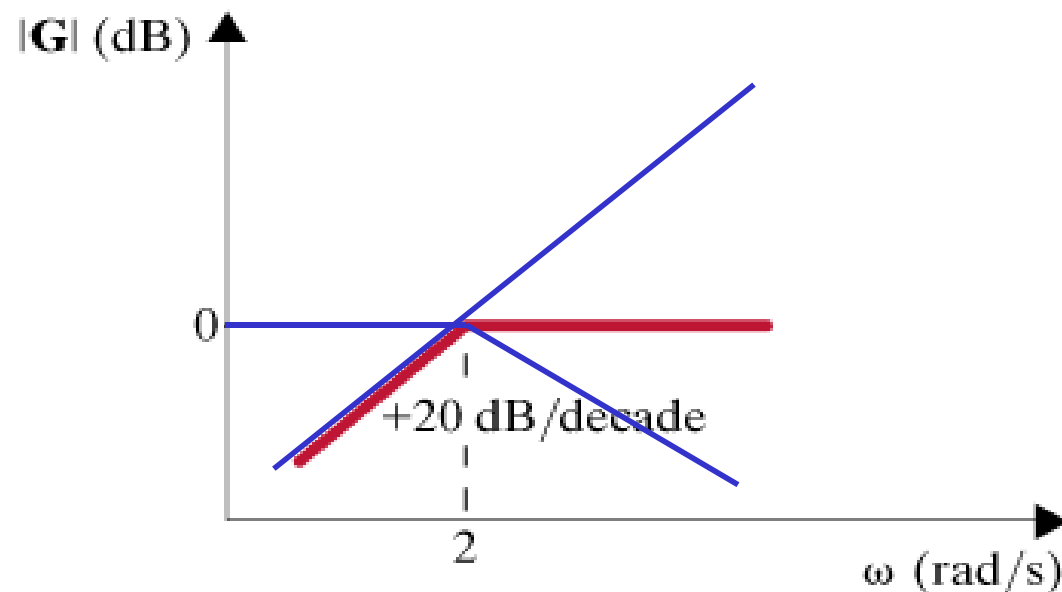
20dB/dec. Crosses 0dB at $\omega = \frac{1}{\tau} = 2 \text{rad/s}$

$$G_v(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC}$$

Break/corner frequency : 2rad/s

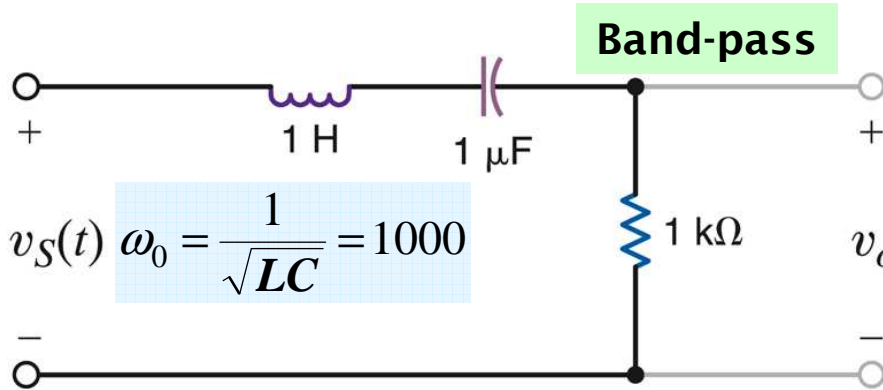
low frequency asymptote of 0dB/dec

High frequency asymptote of -20dB/dec



LEARNING EXTENSION

Sketch the magnitude characteristic of the Bode plot for $G_v(j\omega)$



20dB/dec. Crosses 0dB at $\omega = \frac{1}{RC} = 1000 \text{ rad/s}$

$$G_v(j\omega) = \frac{R}{R + \frac{1}{j\omega C} + j\omega L} = \frac{j\omega RC}{1 + j\omega RC + (j\omega)^2 LC}$$

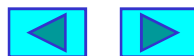
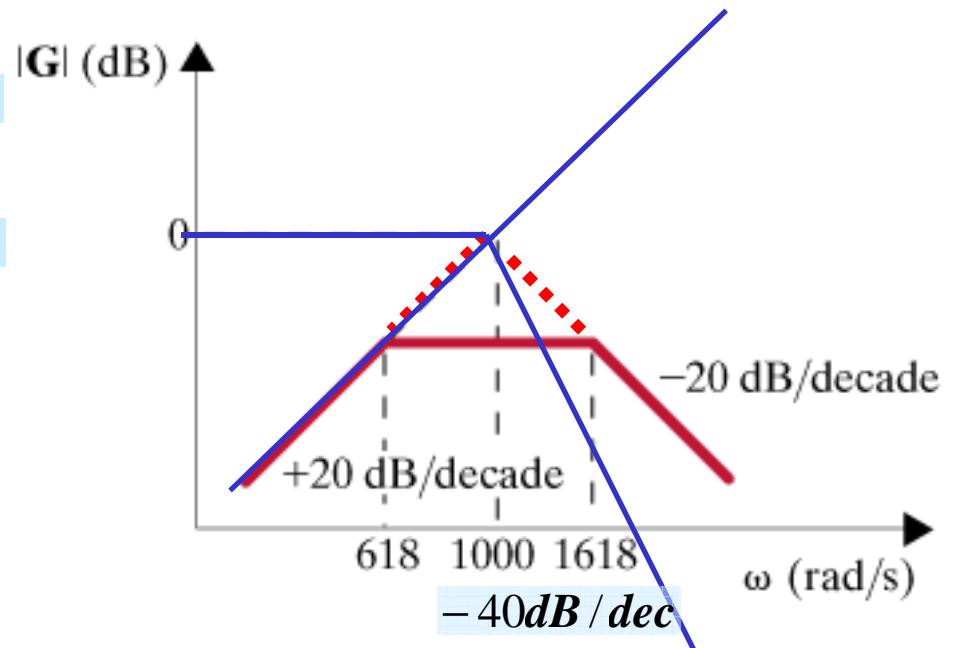
$$\tau^2 = LC \Rightarrow \tau = \sqrt{10^{-6}} = 10^{-3},$$

$$2\zeta\tau = RC = 10^3 \times 10^{-6} \Rightarrow \zeta = \frac{10^{-3}}{2 \times 10^{-3}} = 0.5$$

Break/corner frequency : 1000 rad/s
 low frequency asymptote of 0dB/dec
 High frequency asymptote of -40dB/dec

$$\omega_{LO} = \frac{-(R/L) + \sqrt{(R/L)^2 + 4\omega_0^2}}{2} = 618 \text{ rad/s}$$

$$\omega_{HI} = \frac{(R/L) + \sqrt{(R/L)^2 + 4\omega_0^2}}{2} = 1618 \text{ rad/s}$$



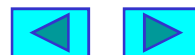
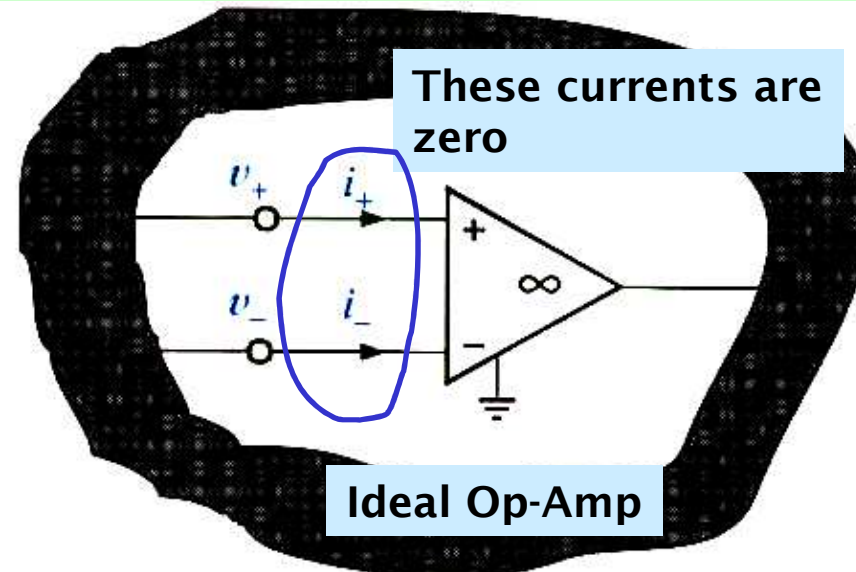
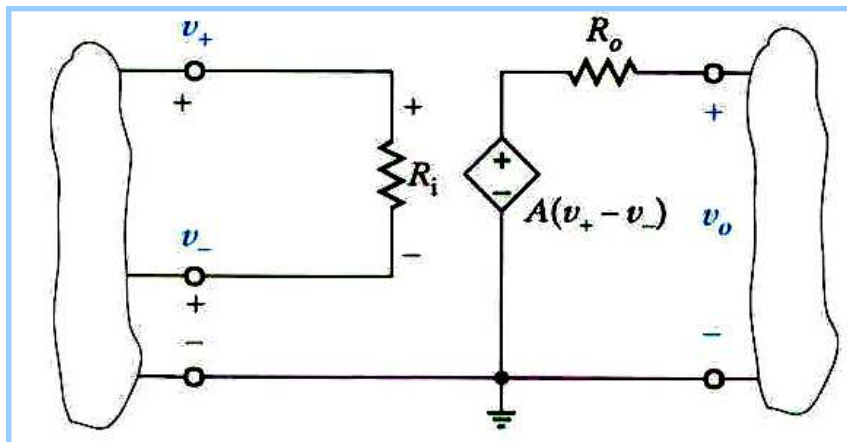
ACTIVE FILTERS

Passive filters have several limitations

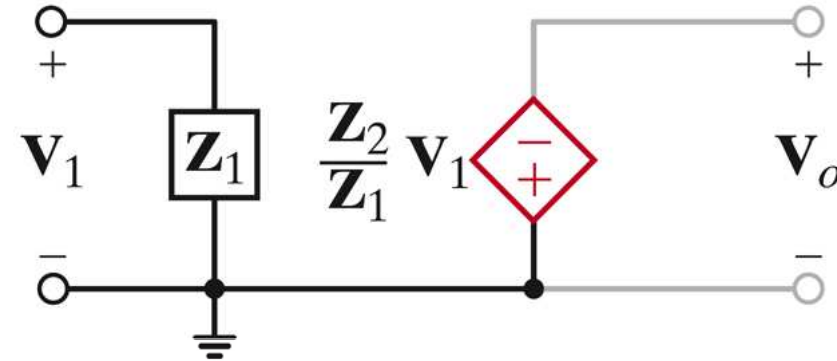
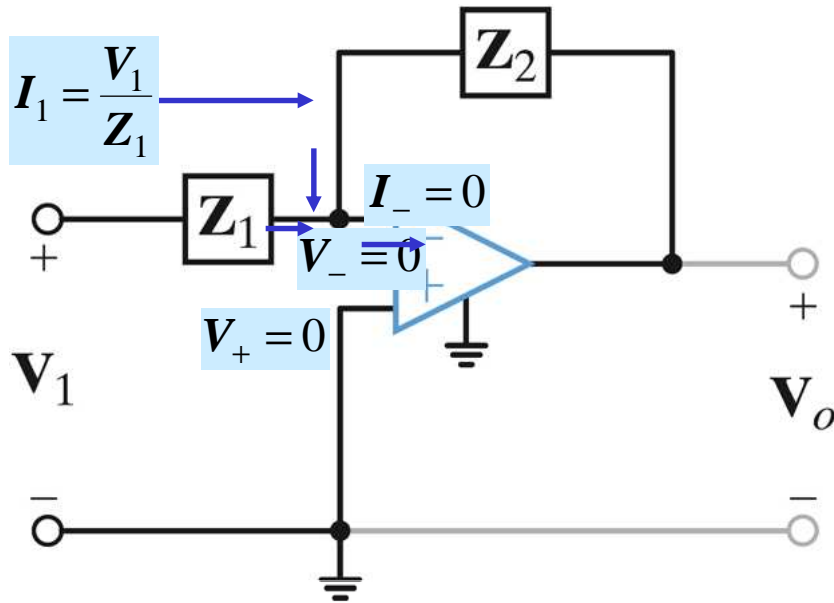
1. Cannot generate gains greater than one
2. Loading effect makes them difficult to interconnect
3. Use of inductance makes them difficult to handle

Using operational amplifiers one can design all basic filters, and more, with only resistors and capacitors

The linear models developed for operational amplifiers circuits are valid, in a more general framework, if one replaces the resistors by impedances



Basic Inverting Amplifier



Infinite gain $\Rightarrow V_- = V_+$

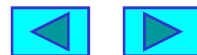
Infinite input impedance $\Rightarrow I_- = I_+ = 0$

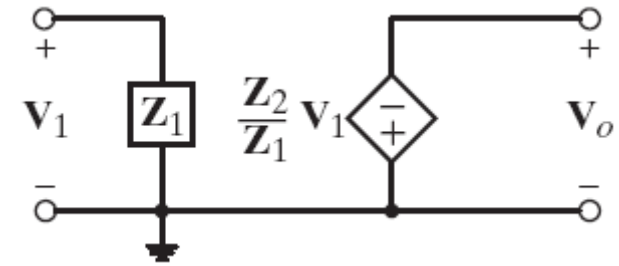
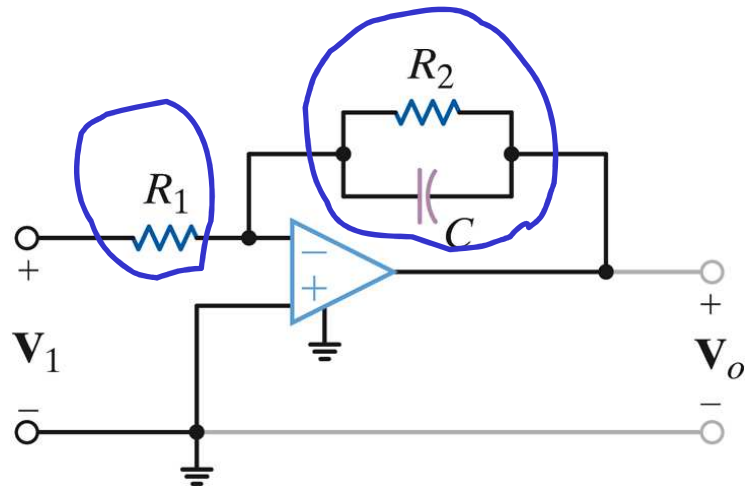
$$\frac{V_1}{Z_1} + \frac{V_o}{Z_2} = 0$$

$$V_o = -\frac{Z_2}{Z_1} V_1$$

$$G = -\frac{Z_2}{Z_1}$$

Linear circuit equivalent



EXAMPLE**USING INVERTING AMPLIFIER**

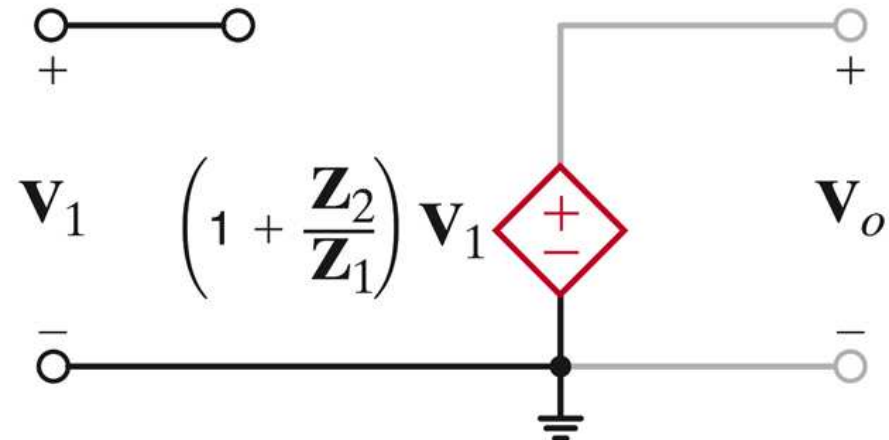
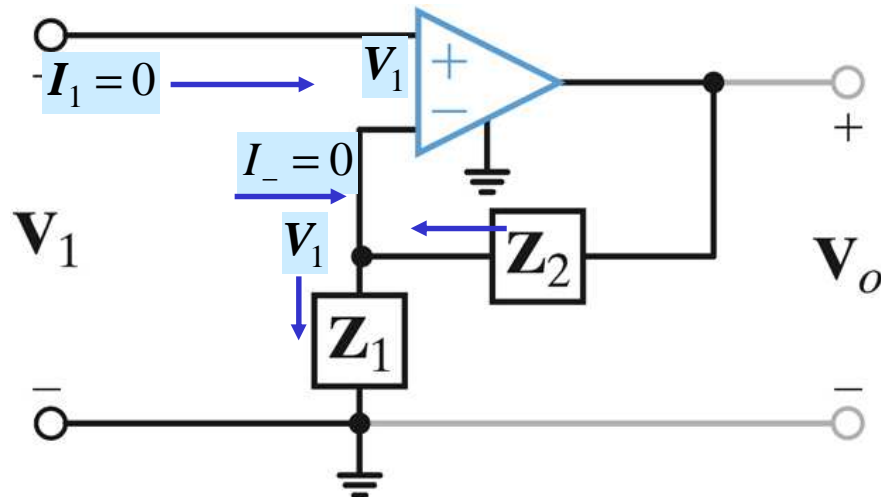
$$Z_1 = R_1$$

$$Z_2 = \frac{R_2/j\omega C}{R_2 + 1/j\omega C} = \frac{R_2}{j\omega R_2 C + 1}$$

$$G_v(j\omega) = \frac{V_o(j\omega)}{V_1(j\omega)} = \frac{-R_2/R_1}{j\omega R_2 C + 1}$$

LOW PASS FILTER

Basic Non-inverting amplifier



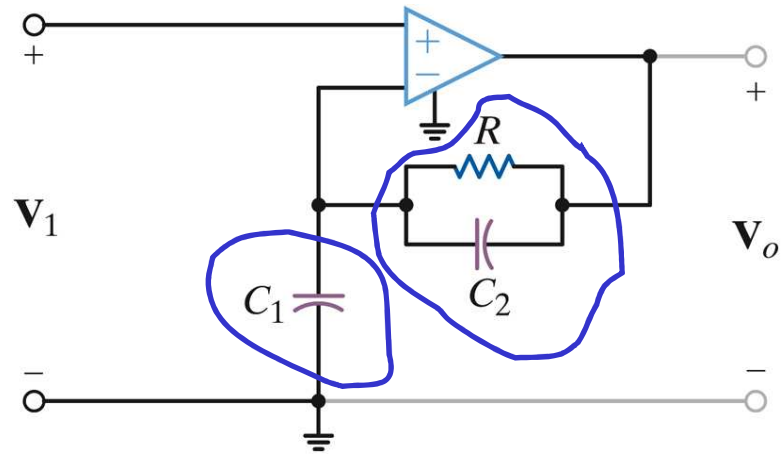
$$\frac{V_0 - V_1}{Z_2} = \frac{V_1}{Z_1}$$

$$V_0 = \frac{Z_2 + Z_1}{Z_1} V_1$$

$$G = 1 + \frac{Z_2}{Z_1}$$



EXAMPLE USING NON INVERTING CONFIGURATION

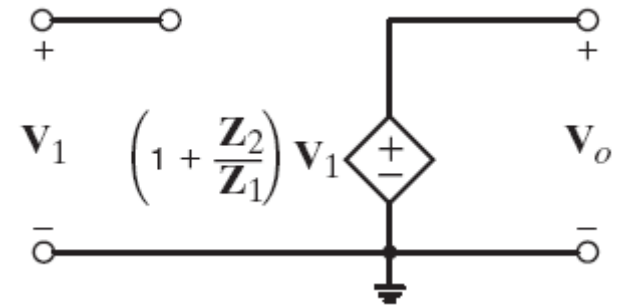


$$G_v(j\omega) = \frac{V_o(j\omega)}{V_1(j\omega)} = 1 + \frac{R/(j\omega RC_2 + 1)}{1/j\omega C_1}$$

$$= \frac{j\omega(RC_1 + RC_2) + 1}{j\omega RC_2 + 1}$$

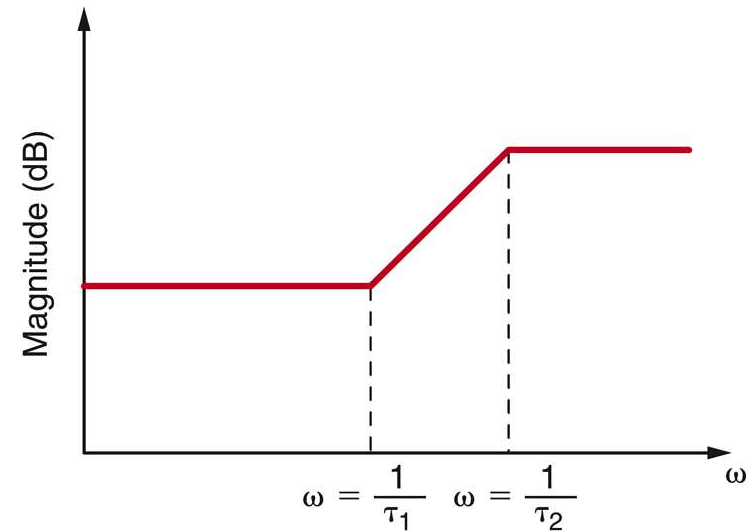
$$= \frac{j\omega\tau_1 + 1}{j\omega\tau_2 + 1}$$

$$\tau_1 = R(C_1 + C_2) \text{ and } \tau_2 = RC_2.$$



$$Z_1 = \frac{1}{j\omega C_1}$$

$$Z_2 = \frac{R}{j\omega RC_2 + 1}$$

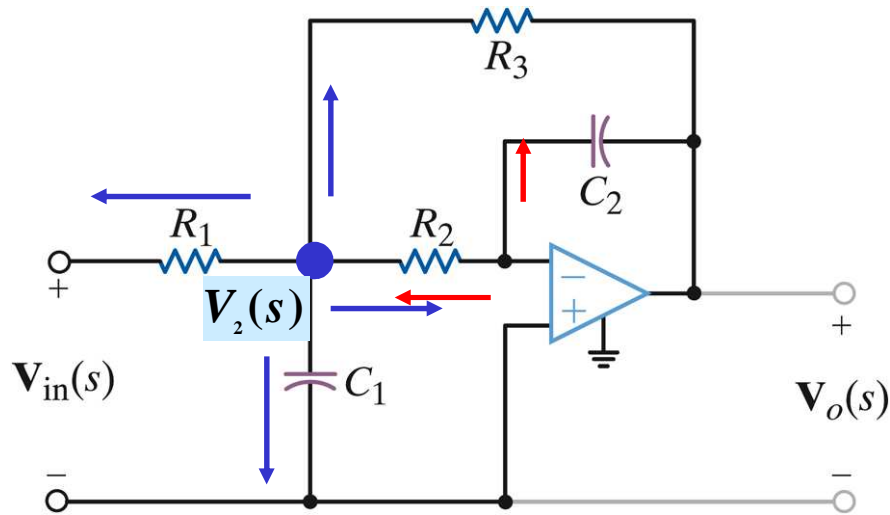


(b)



EXAMPLE

SECON ORDER FILTER



$$\frac{V_2 - V_{IN}}{R_1} + \frac{V_2}{1/C_1 s} + \frac{V_2}{R_2} + \frac{V_2 - V_o}{R_3} = 0$$

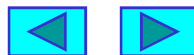
$$-\frac{V_2}{R_2} - \frac{V_o}{1/C_2 s} = 0$$

$$\omega_c = \frac{1}{\sqrt{R_3 R_2 C_1 C_2}}$$

$$H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{-\left(\frac{R_3}{R_1}\right)\left(\frac{1}{R_3 R_2 C_1 C_2}\right)}{s^2 + s\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} + \frac{1}{R_3 C_1}\right) + \frac{1}{R_3 R_2 C_1 C_2}}$$

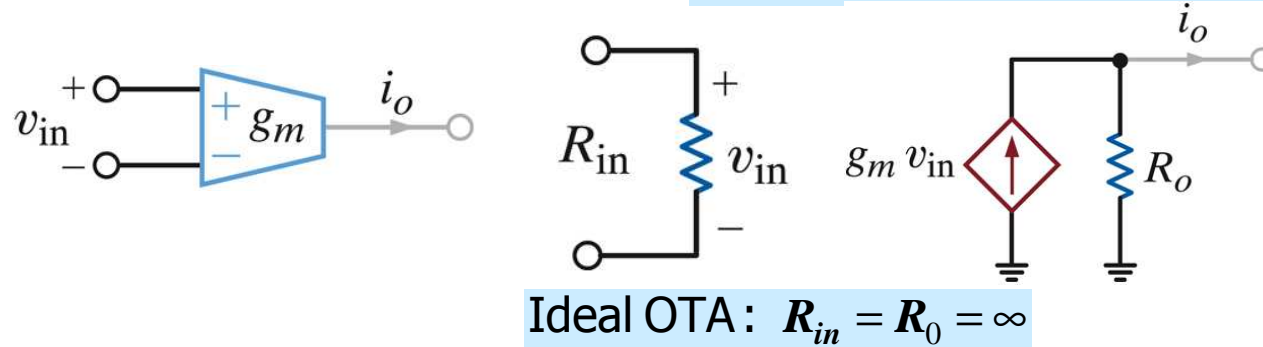
$$2\zeta\omega_c = \frac{1}{C_1} \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right)$$

$$\zeta = \frac{1}{2} \sqrt{\frac{C_2}{C_1}} \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) \sqrt{R_2 R_3}$$

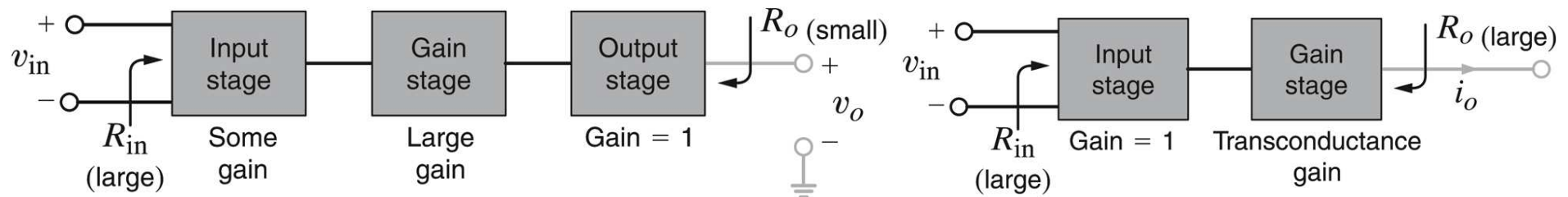


Operational Transconductance Amplifier (OTA)

Due to the internal op-amp circuitry, it has limitations, e.g., for high frequency and/or low voltage situations. The *Operational Transconductance Amplifier* (OTA) performs well in those situations



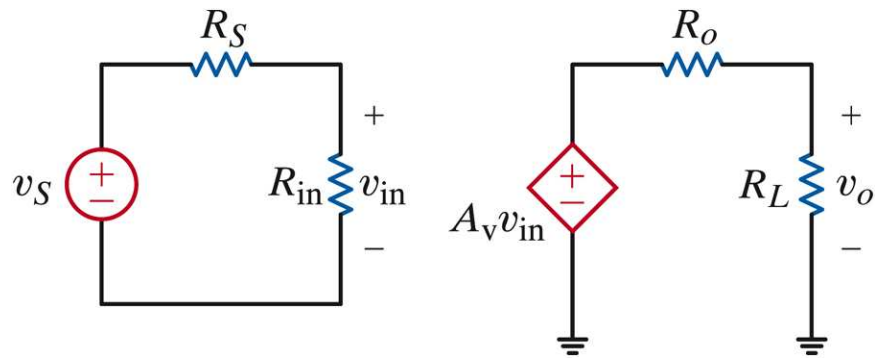
COMPARISON BETWEEN OP-AMPS AND OTAs - PHYSICAL CONSTRUCTION



Comparison of Op-Amp and OTA - Parameters

Amplifier Type	Ideal R_{in}	Ideal R_o	Ideal Gain	Input Current	input Voltage
Op-Amp	∞	0	∞	0	0
OTA	∞	∞	g_m	0	nonzero





Basic Op-Amp Circuit

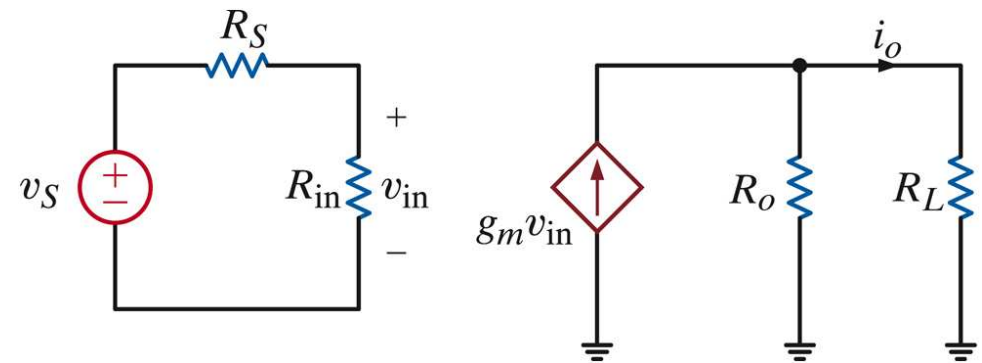
$$v_o = \frac{R_L}{R_o + R_L} A_v v_{in}$$

$$v_{in} = \frac{R_{in}}{R_S + R_{in}} V_S$$

$$A = \frac{v_o}{V_S} = \left[\frac{R_L}{R_o + R_L} \right] A_v \left[\frac{R_{in}}{R_S + R_{in}} \right]$$

Ideal Op - Amp

$$A = A_v = \infty$$



Basic OTA Circuit

$$i_o = \frac{R_o}{R_o + R_L} g_m v_{in}$$

$$v_{in} = \frac{R_{in}}{R_S + R_{in}} V_S$$

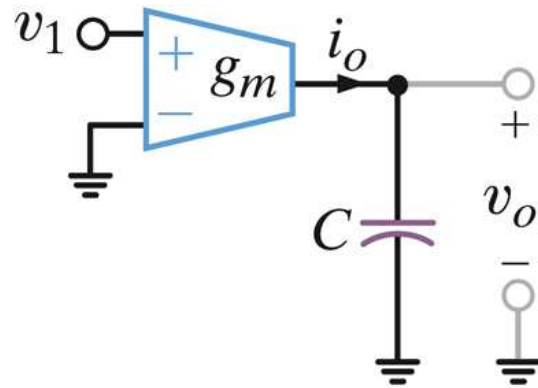
$$G_m = \frac{i_o}{v_{in}} = \left[\frac{R_o}{R_o + R_L} \right] g_m \left[\frac{R_{in}}{R_S + R_{in}} \right]$$

Ideal OTA

$$G_m = g_m$$



Basic OTA Circuits



$$i_o = g_m v_1$$

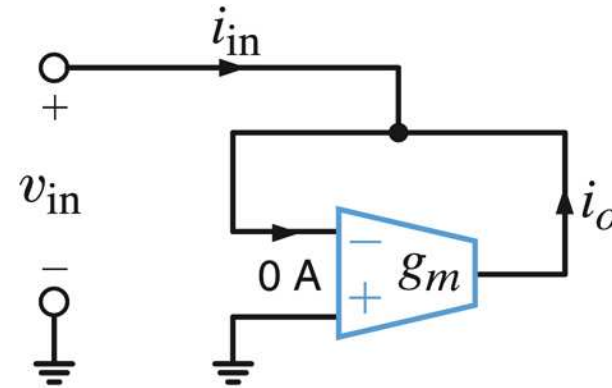
$$v_o = \frac{1}{C} \int_0^t i_o(x) dx + v_o(0)$$

Integrator

$$v_o(t) = \frac{g_m}{C} \int_0^t v_1(x) dx + v_o(0)$$

In the frequency domain

$$V_o = \frac{g_m}{j\omega C} V_1$$

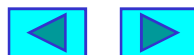


$$i_o = -g_m v_{in} \quad (\text{notice polarity})$$

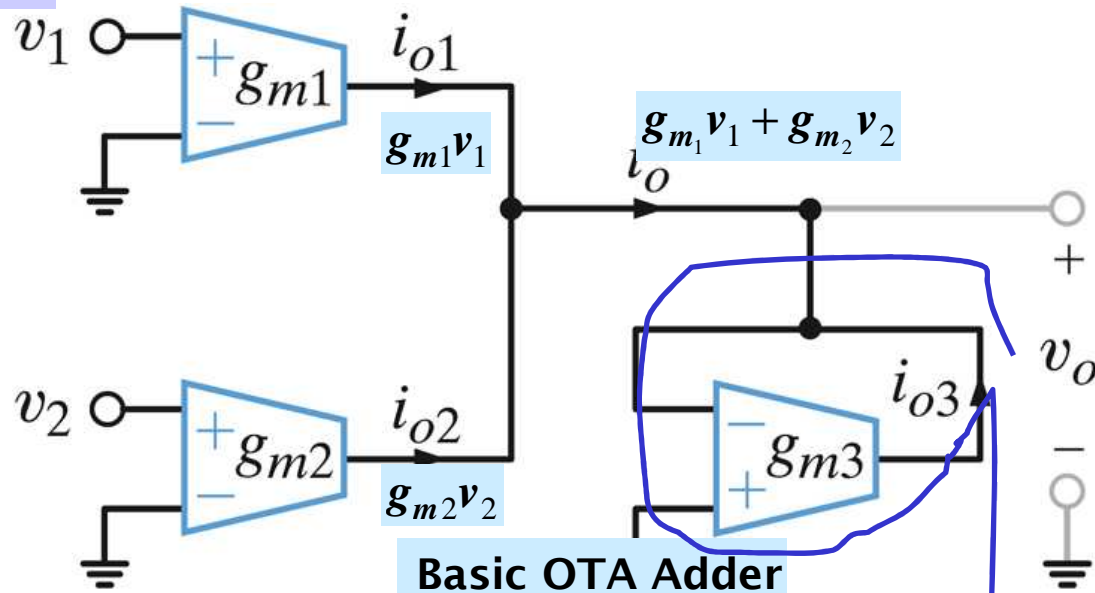
$$i_{in} + i_o = 0$$

Simulated Resistor

$$\frac{v_{in}}{i_{in}} = R_{eq} = \frac{1}{g_m}$$



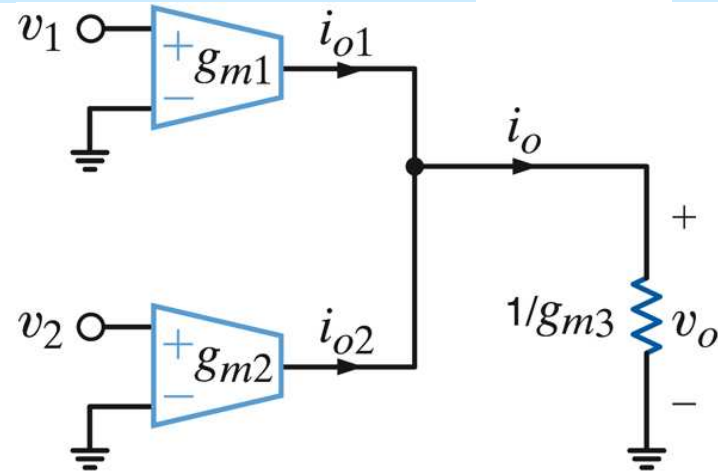
OTA APPLICATION



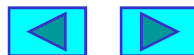
Basic OTA Adder

$$v_o = \frac{1}{g_{m3}} (g_{m1}v_1 + g_{m2}v_2)$$

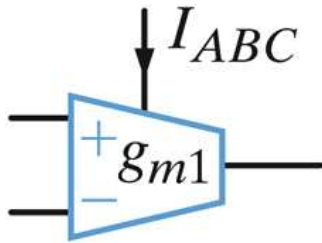
Simulated Resistor



Equivalent representation



Programmability of g_m



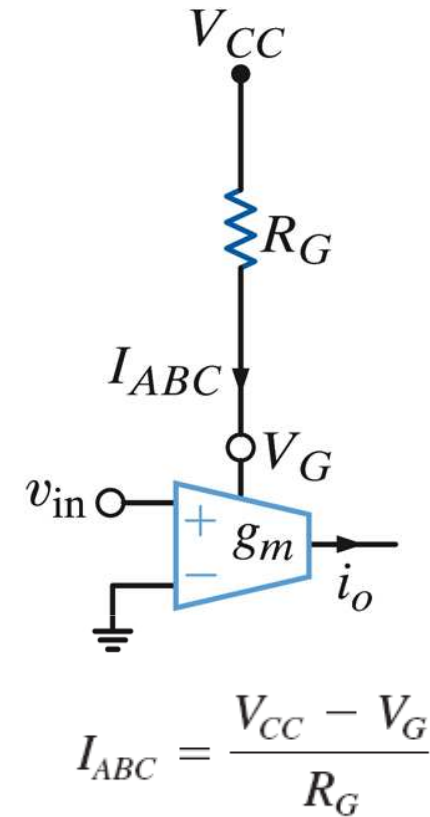
$$g_{m1} = \left[20 \frac{S}{A} \right] I_{ABC}$$

Typical values

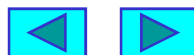
$$g_m \leq 10mS$$

g_m range: 3 - 7 decades

$$(e.g., g_m \geq \frac{10mS}{10^7})$$

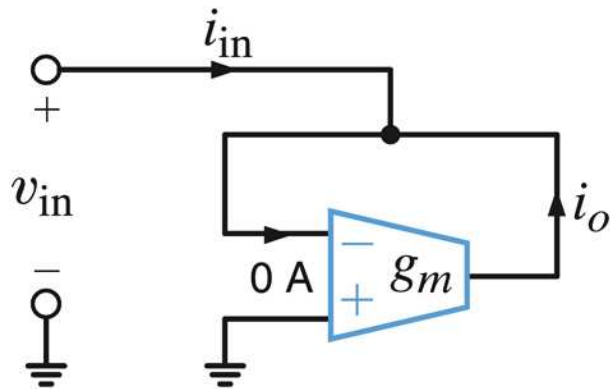


Controlling transconductance



LEARNING EXAMPLE

Produce a $25k\Omega$ resistor



$$\begin{aligned}g_m &\leq 4\text{mS} \\g_m &\geq \frac{4\text{mS}}{10^4} = 4 \times 10^{-7}\text{ S} \\g_m &= 20I_{ABC}\end{aligned}$$

Simulated Resistor

$$\frac{v_{in}}{i_{in}} = R_{eq} = \frac{1}{g_m}$$

$$25 \times 10^3 = \frac{1}{g_m} \Rightarrow g_m = 4 \times 10^{-5}\text{ S} > 4 \times 10^{-7}\text{ S}$$

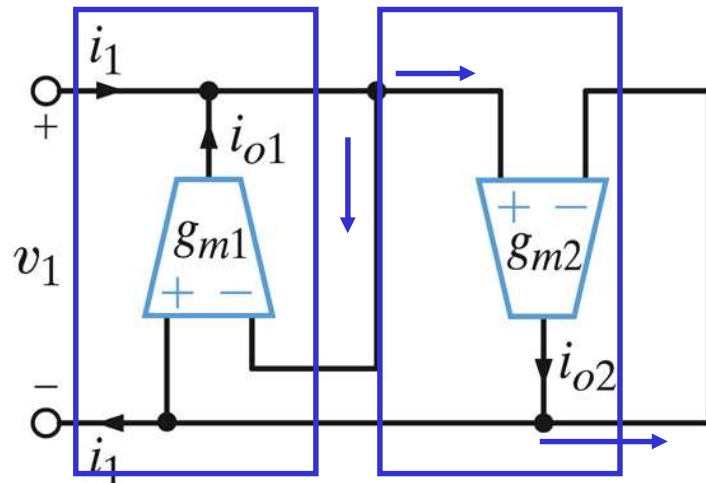
$$4 \times 10^{-5}\text{ S} = 20 \left[\frac{\text{S}}{\text{A}} \right] I_{ABC} (\text{A})$$

$$I_{ABC} = 2 \times 10^{-6}\text{ A} = 2\mu\text{A}$$



LEARNING EXAMPLE

Floating simulated resistor



$$i_{o1} = -g_{m1}v_1$$

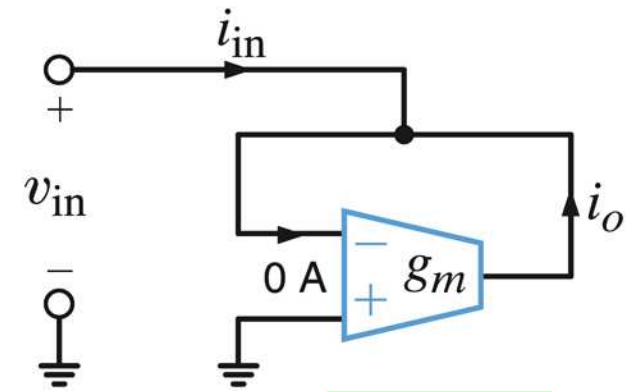
$$i_{o2} = g_{m2}v_1$$

$$i_1 = -i_{o1}$$

$$i_{o2} = i_1$$

For proper operation

$$g_{m1} = g_{m2}$$



$$i_o = -g_m v_{in}$$

One grounded terminal

Produce a 10MΩ resistor

$$g_m \leq 4mS$$

$$g_m \geq \frac{4mS}{10^4} = 4 \times 10^{-7} S$$

$$g_m = 20I_{ABC}$$

$$g_m = \frac{1}{10 \times 10^6} = 10^{-7} S < 4 \times 10^{-7} S$$

The resistor cannot be produced with this OTA!

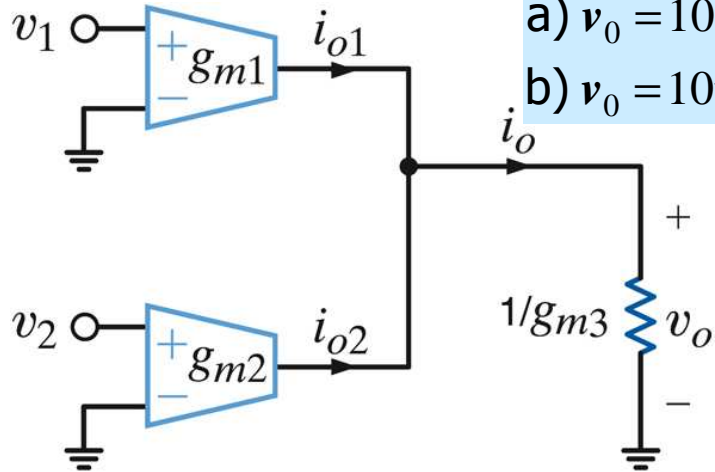


LEARNING EXAMPLE

Select g_{m1} , g_{m2} , g_{m3} , to produce

a) $v_o = 10v_1 + 2v_2$

b) $v_o = 10v_1 - 2v_2$



$g_m \leq 4mS$

$g_m \geq \frac{4mS}{10^4} = 4 \times 10^{-7} S$

$g_m = 20I_{ABC}$

$$v_o = \frac{1}{g_{m3}} (g_{m1}v_1 + g_{m2}v_2)$$

Case a

$\frac{g_{m1}}{g_{m3}} = 10; \frac{g_{m2}}{g_{m3}} = 2$

Two equations in three unknowns.
Select one transconductance

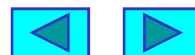
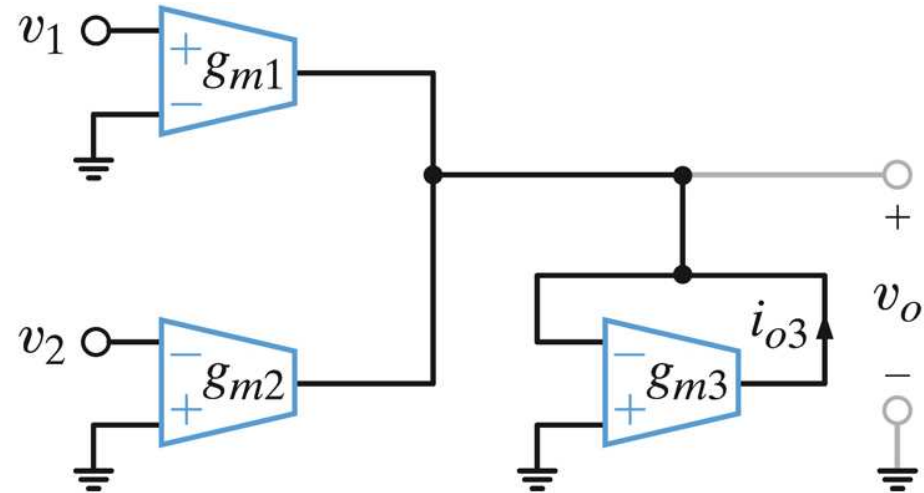
$g_{m3} = 0.1mS \Rightarrow I_{ABC3} = \frac{1}{20} \times 10^{-4} (A) = 5\mu A$

$g_{m2} = 0.2mS \Rightarrow I_{ABC2} = 10\mu A$

$g_{m1} = 1mS \Rightarrow I_{ABC1} = 50\mu A$

Case b

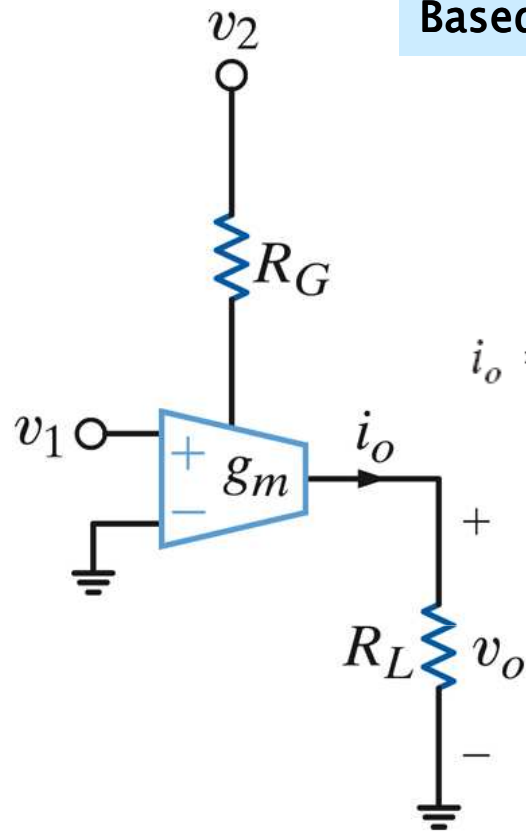
Reverse polarity of v2!



ANALOG MULTIPLIER

Based on 'modulating the control current'

ASSUMES V_G IS ZERO

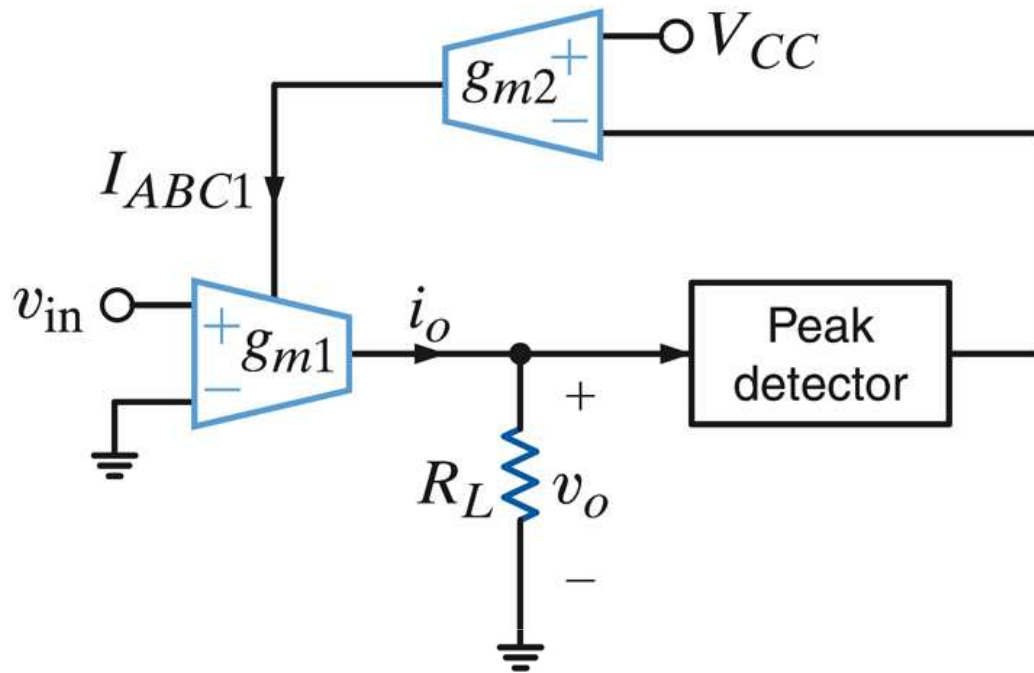


$$i_o = v_1 g_m = v_1 (20 I_{ABC}) = 20 v_1 \left[\frac{v_2}{R_G} \right]$$

$$v_o = i_o R_L = 20 \left[\frac{R_L}{R_G} \right] v_1 v_2$$



AUTOMATIC GAIN CONTROL



$$v_o = i_o R_L = v_{in} g_{m1} R_L = 20 v_{in} R_L g_{m2} (V_{CC} - |v_o|) = A v_{in} - B v_{in} |v_o|$$

$$v_{in} g_{m1} R_L = v_{in} R_L (20 I_{ABC1})$$

$$v_o = \frac{A v_{in}}{1 + B v_{in}}$$

For simplicity of analysis we drop the absolute value

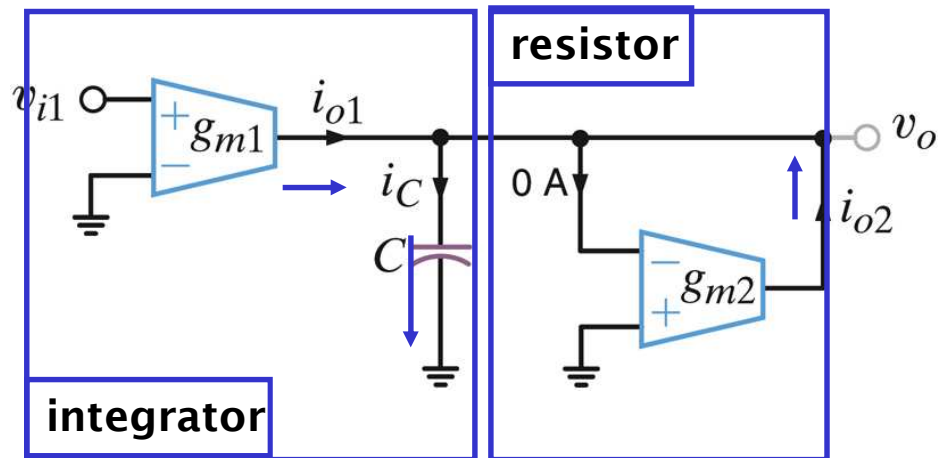
$$v_{IN} \text{ small} \Rightarrow v_o \approx A v_{IN}$$

$$v_{IN} \text{ big} \Rightarrow v_o \approx \frac{A}{B}$$

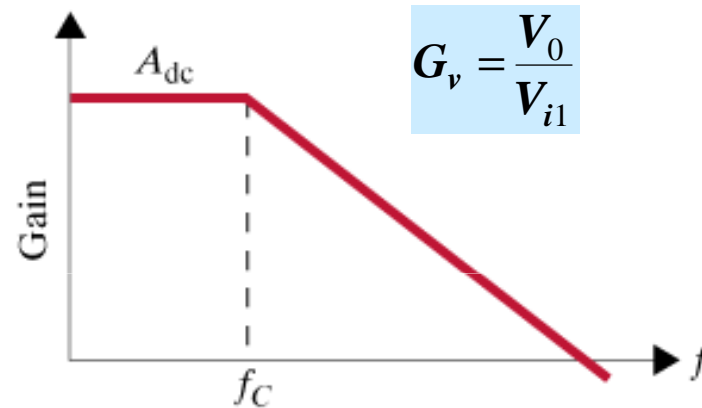


OTA-C CIRCUITS

Circuits created using capacitors, simulated resistors, adders and integrators



Magnitude Bode plot



Frequency domain analysis assuming ideal OTAs

$$V_0 = \frac{1}{j\omega C} I_C \quad I_C = I_{o1} + I_{o2}$$

$$I_{o1} = g_{m1} V_{i1} \quad I_{o2} = -g_{m2} V_0$$

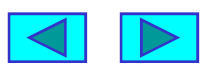
$$V_0 = \frac{1}{j\omega C} [g_{m1} V_{i1} - g_{m2} V_0]$$

$$V_0 = \frac{g_{m1} / g_{m2}}{1 + j\omega C / g_{m2}} V_{i1}$$

$$A_{dc} = \frac{g_{m1}}{g_{m2}}$$

$$\omega_C = \frac{g_{m2}}{C}$$

$$2\pi f_C = \frac{g_{m2}}{C}$$



LEARNING EXAMPLE

Desired: $G_v = \frac{V_o}{V_{i1}} = \frac{4}{1 + \frac{j\omega}{2\pi(10^5)}}$

$g_m \leq 1mS$

$g_m \geq \frac{1mS}{10^3} = 10^{-6} S$

OK

$g_m = 20I_{ABC}$

Find the transconductances and biases

$A_{dc} = \frac{g_{m1}}{g_{m2}}$

$\omega_C = \frac{g_{m2}}{C}$

$2\pi f_C = \frac{g_{m2}}{C}$

$A_{dc} = 4$

$\omega_C = 2\pi(10^5) \Rightarrow f_C = 100kHz$

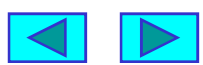
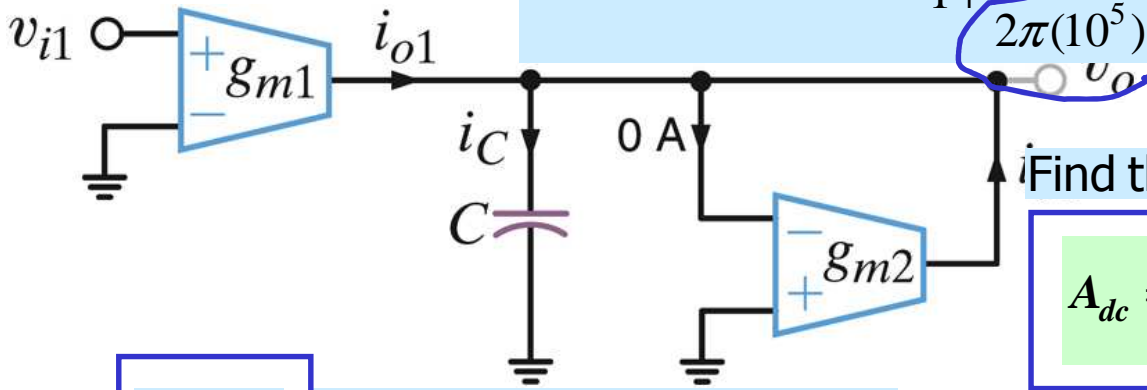
Two equations in three unknowns.
Select the capacitor value

$C = 25pF$

$g_{m2} = 2\pi(10^5)(25 \times 10^{-12}) = 15.7 \times 10^{-6} S$

$I_{ABC2} = \frac{15.7}{20} = 0.785\mu A$

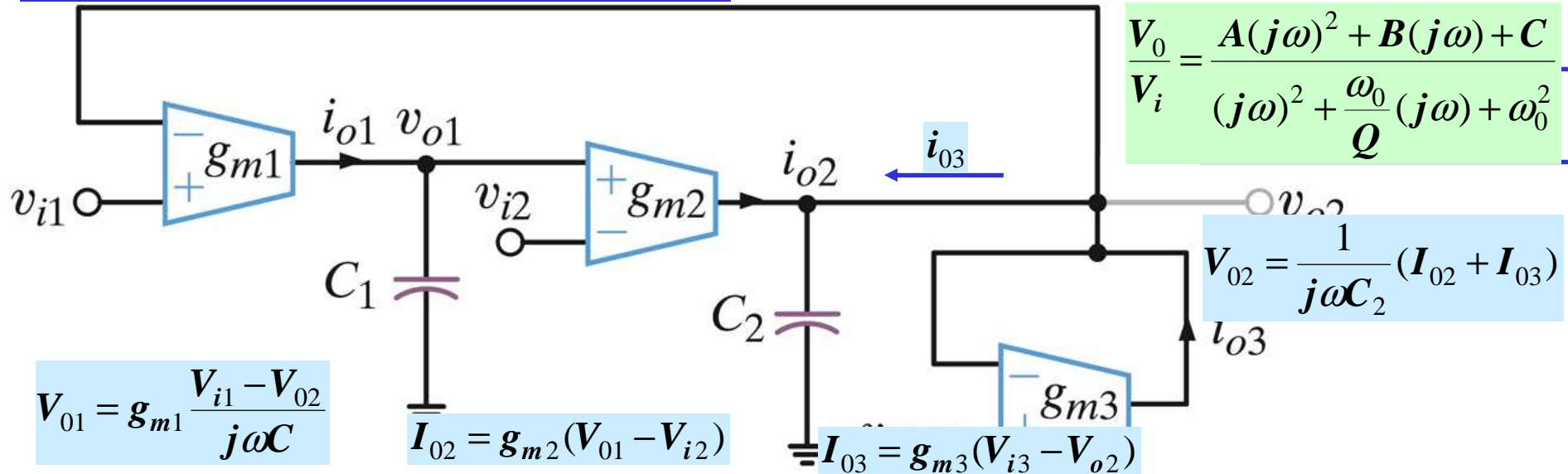
$g_{m1} = 62.8\mu S \Rightarrow I_{ABC1} = 3.14\mu A$



TOW-THOMAS OTA-C BIQUAD FILTER

biquad ~ biquadratic

$$\frac{V_o}{V_i} = \frac{A(j\omega)^2 + B(j\omega) + C}{(j\omega)^2 + \frac{\omega_0}{Q}(j\omega) + \omega_0^2}$$



$$V_{o1} = g_{m1} \frac{V_{i1} - V_{o2}}{j\omega C_1}$$

$$I_{o2} = g_{m2}(V_{o1} - V_{i2})$$

$$I_{o3} = g_{m3}(V_{i3} - V_{o2})$$

$$V_{o2} = \frac{1}{j\omega C_2}(I_{o2} + I_{o3})$$

Four equations and four unknowns ($V_{o1}, V_{o2}, I_{o1}, I_{o2}$)

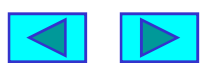
$$V_{o1} = \frac{\left[\frac{j\omega C_2 + g_{m3}}{g_{m2}} V_{i1} + V_{i2} - \frac{g_{m3}}{g_{m2}} V_{i3} \right]}{\left[\frac{C_1 C_2}{g_{m1} g_{m2}} (j\omega)^2 + \frac{g_{m3} C_1}{g_{m2} g_{m1}} (j\omega) + 1 \right]}$$

$$V_{o2} = \frac{V_{i1} - \left[\frac{j\omega C_1}{g_{m1}} V_{i2} + \frac{j\omega C_1 g_{m3}}{g_{m1} g_{m2}} V_{i3} \right]}{\left[\frac{C_1 C_2}{g_{m1} g_{m2}} (j\omega)^2 + \frac{g_{m3} C_1}{g_{m2} g_{m1}} (j\omega) + 1 \right]}$$

$$\omega_0 = \sqrt{\frac{g_{m1} g_{m2}}{C_1 C_2}}, \quad \frac{\omega_0}{Q} = \frac{g_{m3}}{C_2}, \quad Q = \sqrt{\frac{g_{m1} g_{m2}}{g_{m3}^2}} \sqrt{\frac{C_2}{C_1}}$$

$$\left. \begin{matrix} g_{m1} = g_{m2} \\ C_1 = C_2 \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} \omega_0 = \sqrt{g_m / C} \\ Q = \frac{g_m}{g_{m3}} \\ BW = \frac{g_{m3}}{C} \end{matrix} \right.$$

Filter Type	A	B	C
Low-pass	0	0	nonzero
Band-pass	0	nonzero	0
High-pass	nonzero	0	0

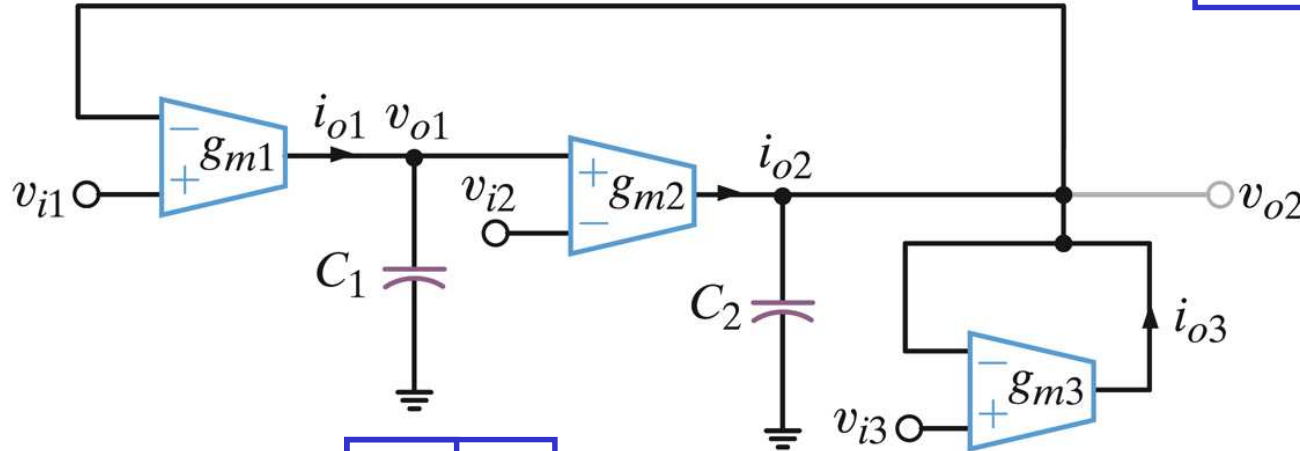


LEARNING EXAMPLE

Design a band-pass filter with center frequency of 500kHz, bandwidth of 75kHz, and center frequency gain -5.

Use the Tow-Thomas configuration and 50-pF capacitors

$$C_1 = C_2$$



$$g_m \leq 4mS$$

$$g_m \geq \frac{4mS}{10^4} = 4 \times 10^{-7} S$$

$$g_m = 20I_{ABC}$$

BW

$$\omega_0 = \sqrt{\frac{g_{m1}g_{m2}}{C_1C_2}}, \quad \frac{\omega_0}{Q} = \frac{g_{m3}}{C_2}, \quad Q = \sqrt{\frac{g_{m1}g_{m2}}{g_{m3}^2}} \sqrt{\frac{C_2}{C_1}}$$

$$V_{i3} = 0$$

$$V_{o2} = \frac{V_{i1} - \left[\frac{j\omega C_1}{g_{m1}} \right] V_{i2} + \left[\frac{j\omega C_1 g_{m3}}{g_{m1}g_{m2}} \right] V_{i3}}{\left[\frac{C_1 C_2}{g_{m1}g_{m2}} \right] (j\omega)^2 + \left[\frac{g_{m3} C_1}{g_{m2}g_{m1}} \right] (j\omega) + 1}$$

$$BW = \frac{g_{m3}}{50 \times 10^{-12}} = 2\pi \times 75 \times 10^3 = 23.56 \mu S$$

$$|G_v(j\omega_0)| = \frac{g_{m2}}{g_{m3}} = 5 \Rightarrow g_{m2} = 117.8 \mu S$$

$$(\omega_0)^2 = (2\pi \times 5 \times 10^5)^2 = \frac{g_{m1} \times 117.8 \times 10^{-6}}{(5 \times 10^{-13})^2} \Rightarrow g_{m1} = 209.5 \mu S$$

$$\omega = \omega_0 \Rightarrow 1 + \left[\frac{C_1 C_2}{g_{m1}g_{m2}} \right] (j\omega)^2 = 0$$

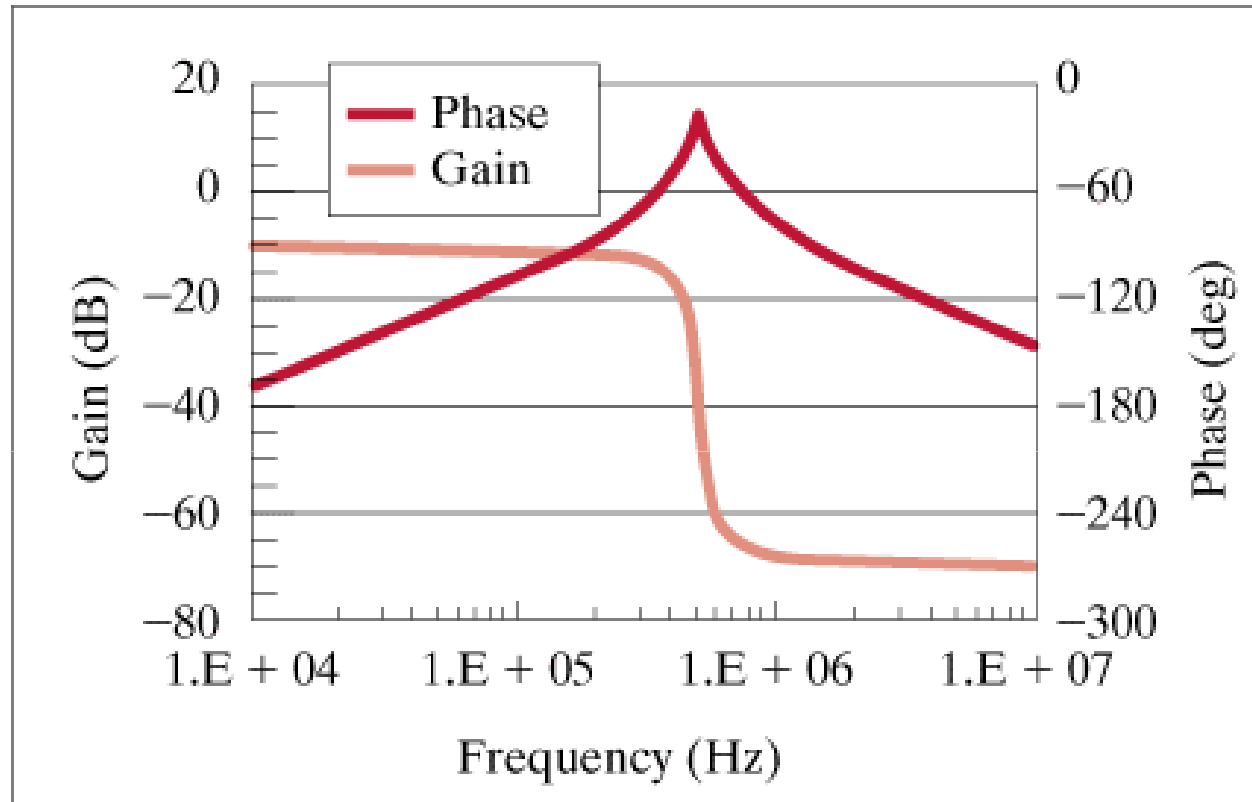
$$I_{ABC1} = 10.47 \mu A$$

$$I_{ABC2} = 5.89 \mu A$$

$$I_{ABC3} = 1.18 \mu A$$

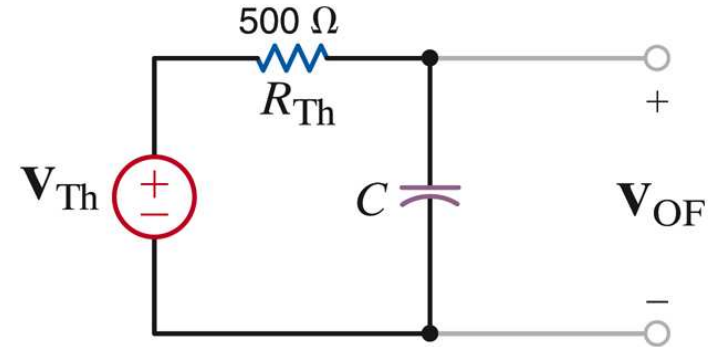
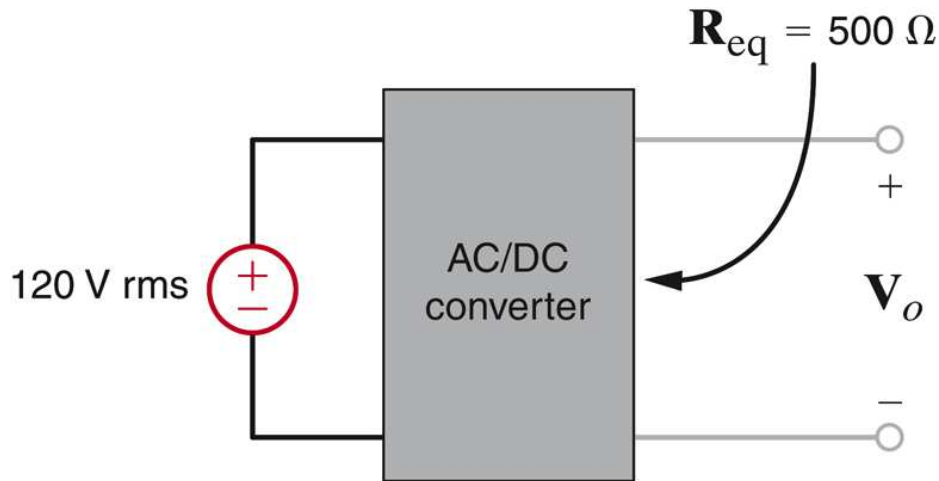


Bode plots for resulting amplifier

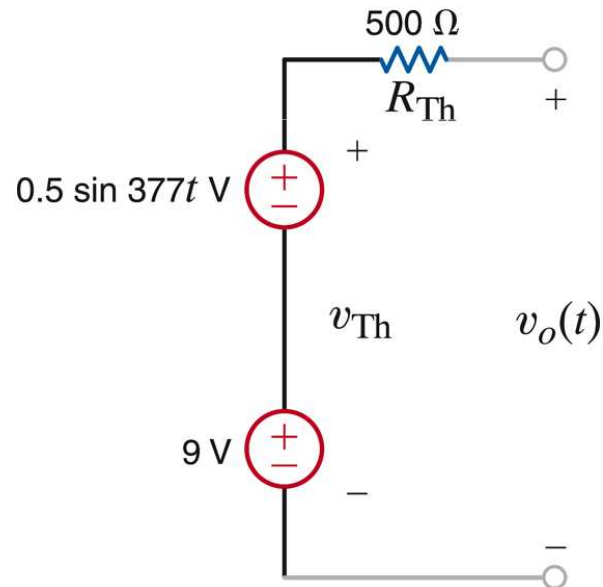


LEARNING BY APPLICATION

Using a low-pass filter to reduce 60Hz ripple



Using a capacitor to create a low-pass filter



Thevenin equivalent for AC/DC converter

$$V_{OF} = \frac{1}{1 + j\omega R_{TH} C} V_{TH}$$

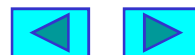
$$|V_{OF}| = \frac{|V_{TH}|}{\sqrt{1 + (\omega R_{TH} C)^2}}$$

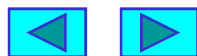
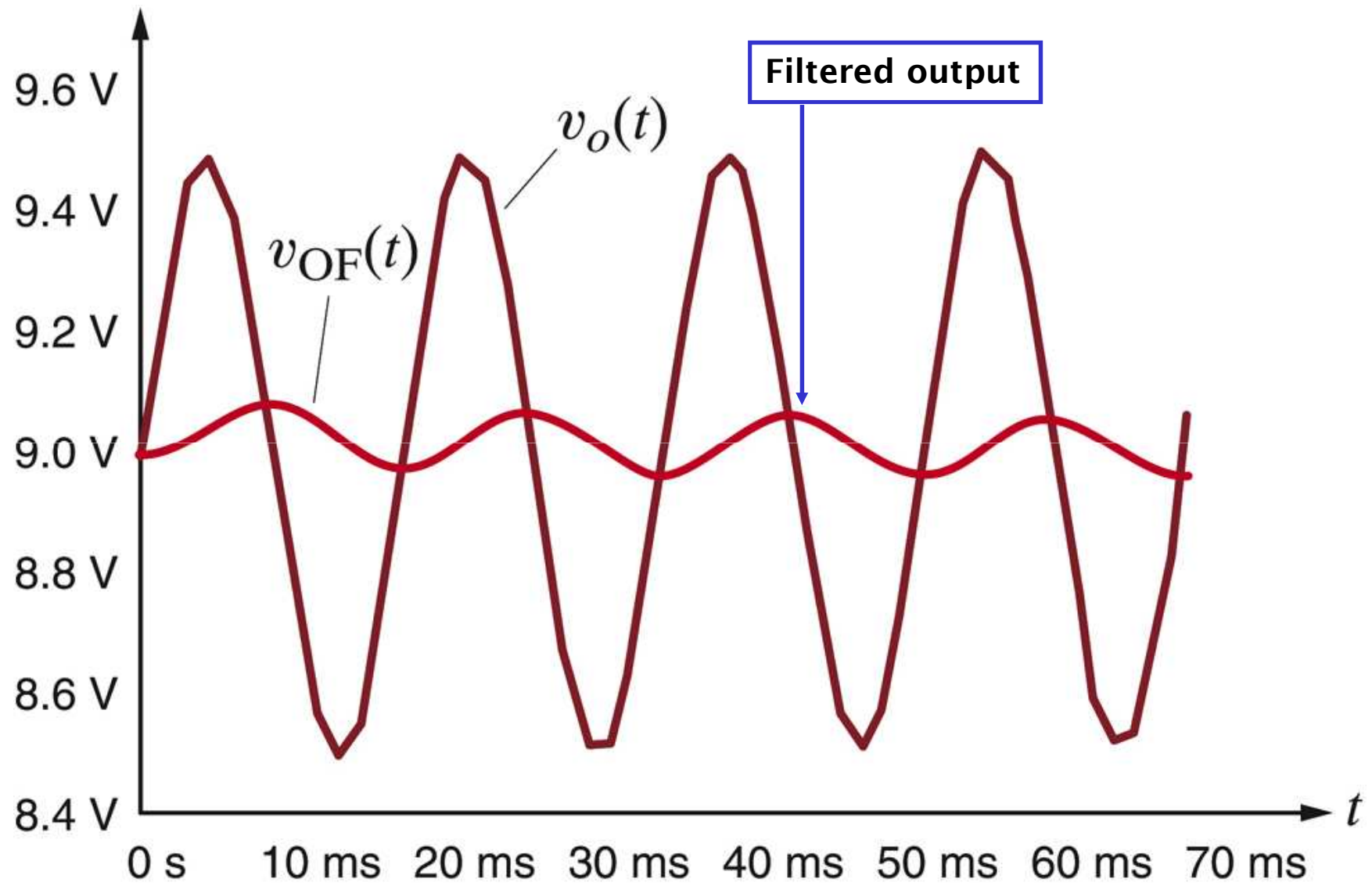
$$\omega_C = \frac{1}{R_{TH} C}$$

Design criterion: place the corner frequency at least a decade lower

$$|V_{OF}| \approx 0.1 |V_{TH}|$$

$$500C = \frac{1}{2\pi \times 6} \Rightarrow C = 53.05 \mu F$$

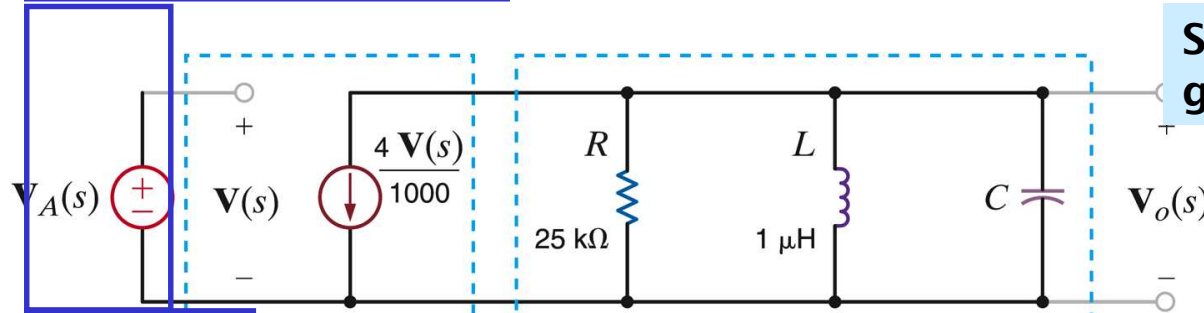




LEARNING EXAMPLE

Single stage tuned transistor amplifier

Select the capacitor for maximum gain at 91.1MHz



Antenna Voltage

Transistor

Parallel resonant circuit

$$\frac{V_o}{V_A} = -\frac{4}{1000} \left[R \parallel j\omega L \parallel \frac{1}{j\omega C} \right]$$

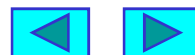
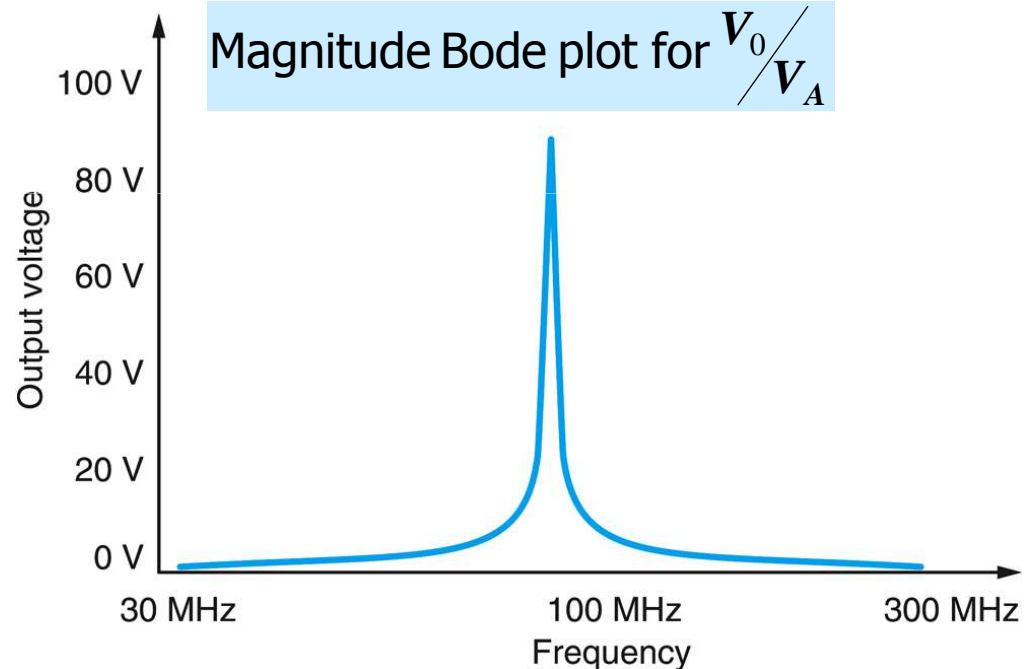
$$= -\frac{4}{1000} \frac{1}{\frac{1}{R} + \frac{1}{j\omega L} + j\omega C} \times \frac{j\omega/C}{j\omega/C}$$

$$\frac{V_o}{V_A} = -\frac{4}{1000} \frac{j\omega/C}{(j\omega)^2 + \frac{j\omega}{RC} + \frac{1}{LC}}$$

Band-pass with center frequency $1/\sqrt{LC}$

$$2\pi(91.1 \times 10^6) = \frac{1}{\sqrt{10^{-6}C}} \Rightarrow C = 3.05 \text{ pF}$$

$$\left| \frac{V_o}{V_A} \right| \left(\omega = \frac{1}{\sqrt{LC}} \right) = \frac{4}{1000} R = 100$$



LEARNING BY DESIGN

Anti-aliasing filter

Nyquist Criterion

When digitizing an analog signal, such as music, any frequency components greater than half the sampling rate will be distorted

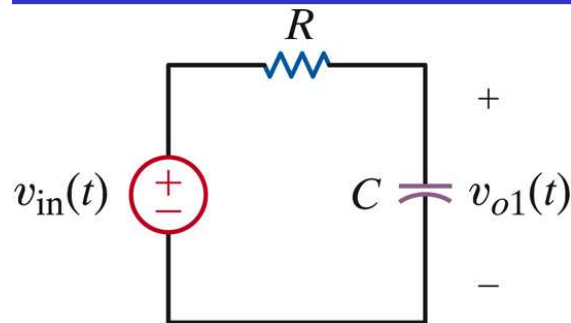
In fact they may appear as spurious components. The phenomenon is known as aliasing.

SOLUTION: Filter the signal before digitizing, and remove all components higher than half the sampling rate. Such a filter is an anti-aliasing filter

For CD recording the industry standard is to sample at 44.1kHz.

An anti-aliasing filter will be a low-pass with cutoff frequency of 22.05kHz

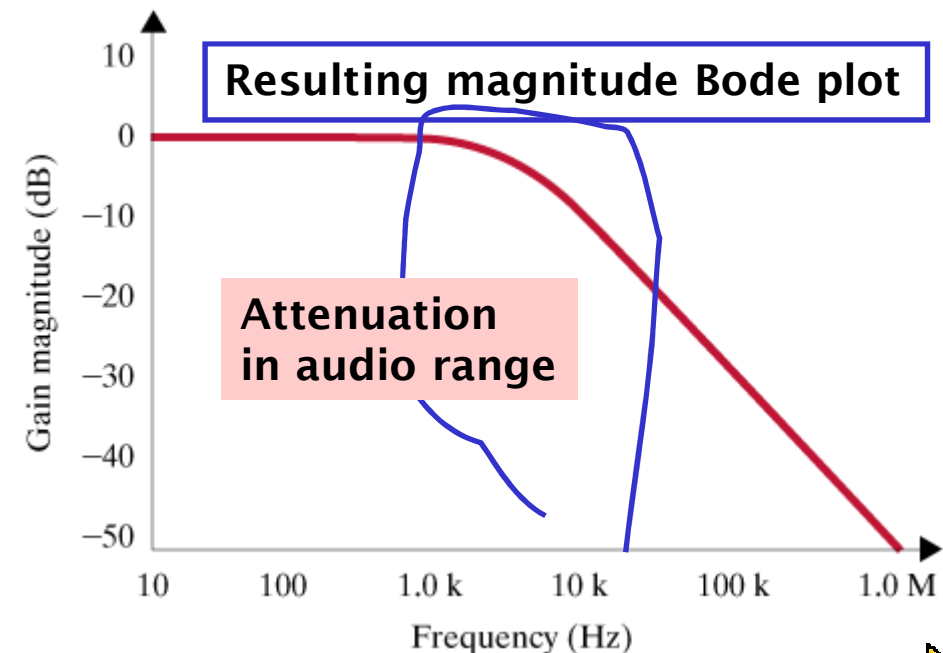
Single-pole low-pass filter



$$\omega_c = \frac{1}{RC} = 2\pi \times 22,050$$

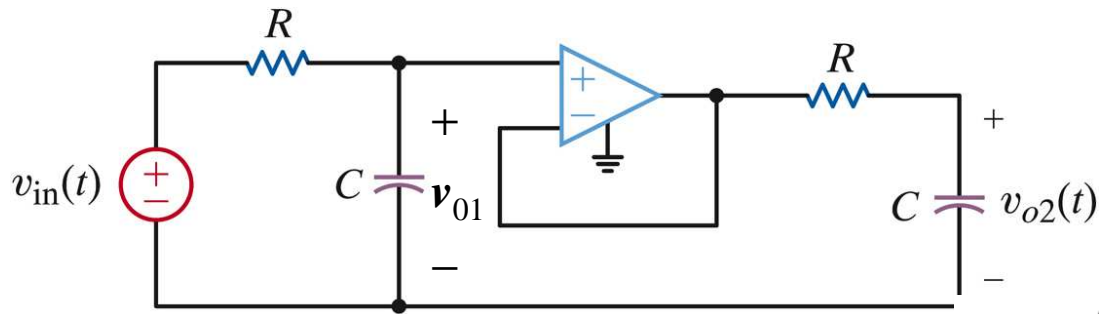
$$C = 1nF \Rightarrow R = 72.18k\Omega$$

$$\frac{V_{o1}}{V_{in}} = \frac{1}{1 + j\omega RC}$$



Improved anti-aliasing filter

Two-stage buffered filter

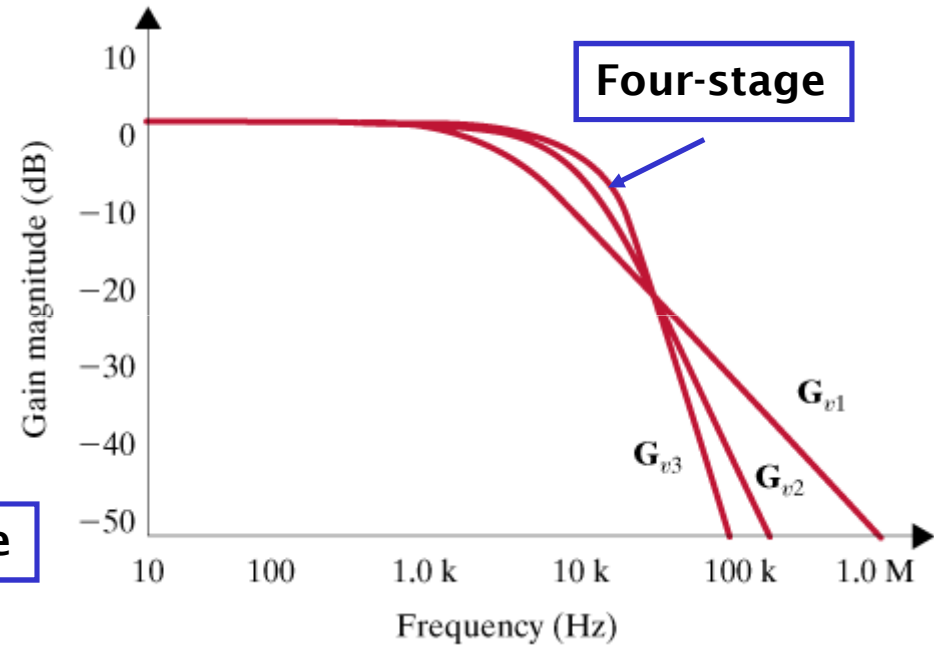
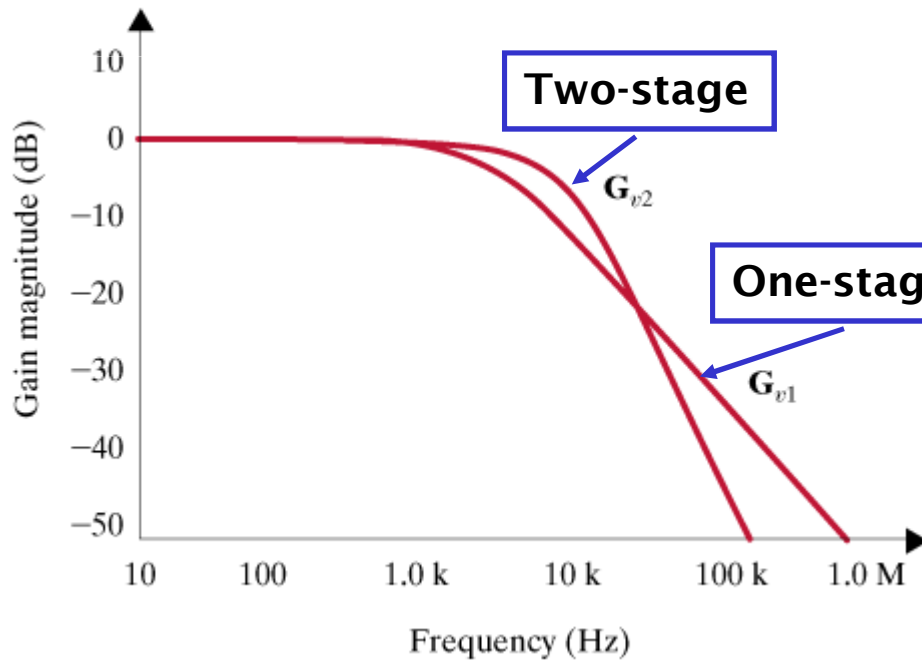


$$\frac{V_{o1}}{V_{in}} = \frac{1}{1 + j\omega RC}$$

$$\frac{V_{o2}}{V_{o1}} = \frac{1}{1 + j\omega RC}$$

n - stage

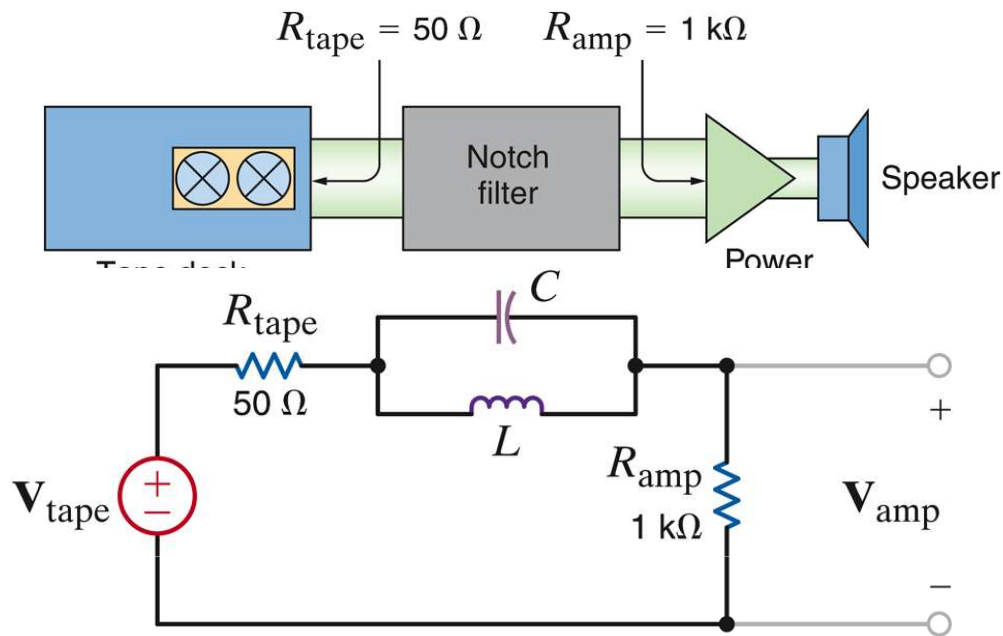
$$\frac{V_{on}}{V_{in}} = \frac{1}{(1 + j\omega RC)^n}$$



LEARNING BY DESIGN

Notch filter to eliminate 60Hz hum

<http://www.wiley.com/college/irwin/0470128690/animations/swf/12-38.swf>

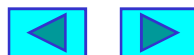
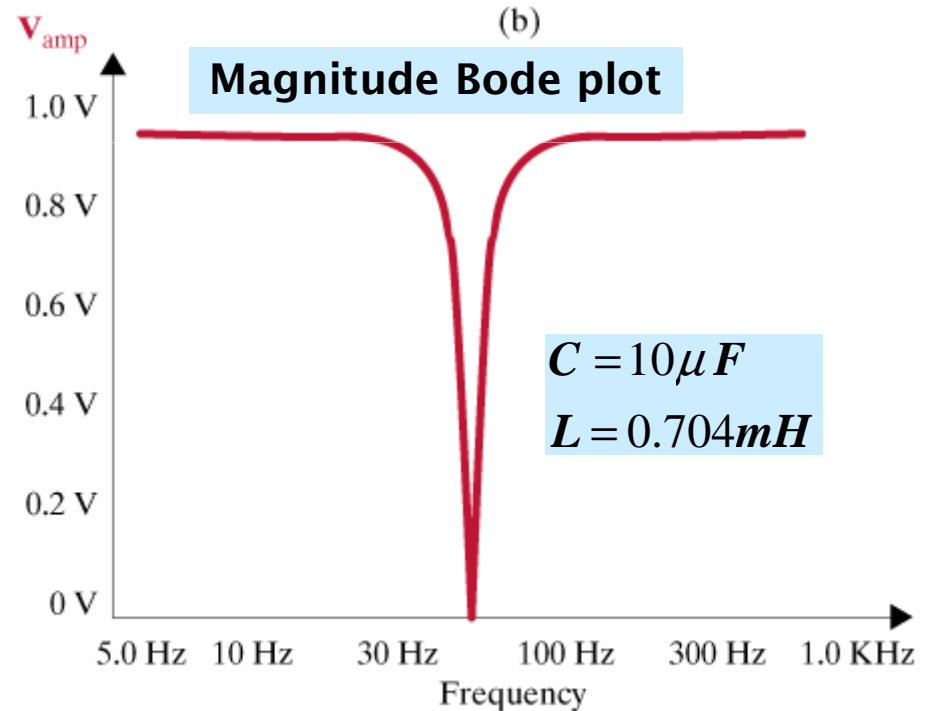
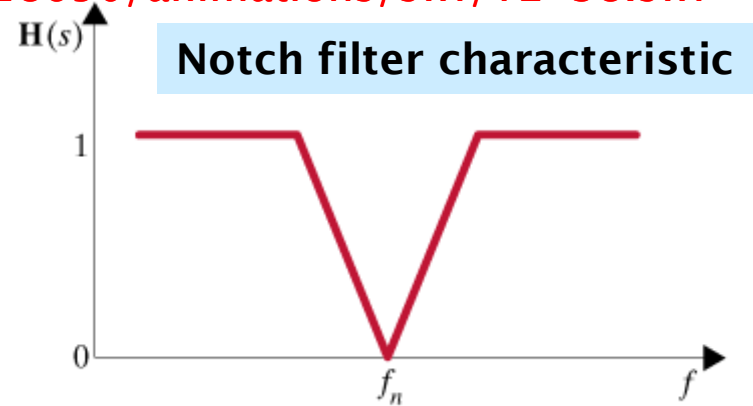


$$\frac{V_{amp}}{V_{tape}} = \frac{R_{amp}}{R_{amp} + R_{tape} + (sL \parallel 1/sC)}$$

$$\frac{V_{amp}}{V_{tape}} = \frac{R_{amp}}{R_{amp} + R_{tape}} \left[\frac{s^2 LC + 1}{s^2 LC + s \left(\frac{L}{R_{amp} + R_{tape}} \right) + 1} \right]$$

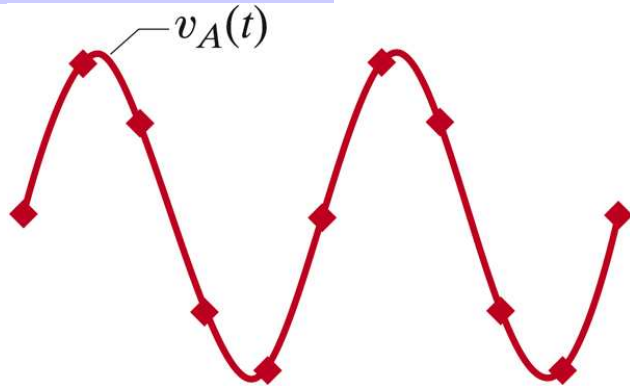
notch frequency = $\frac{1}{\sqrt{LC}}$

To design, pick one, e.g., C and determine the other

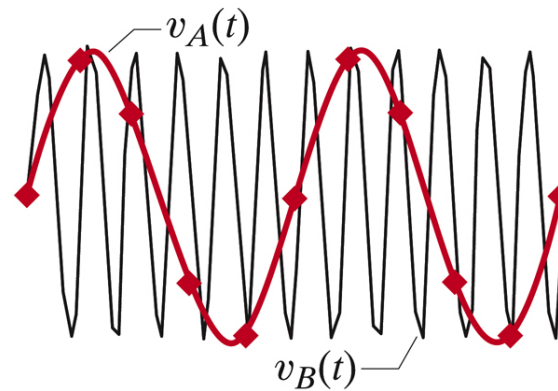


DESIGN EXAMPLE

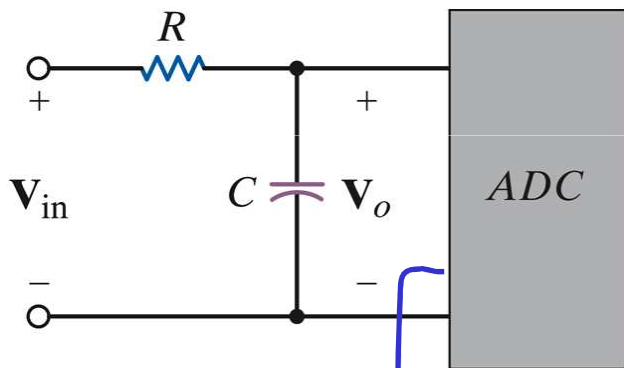
ANTI ALIASING FILTER FOR MIXED MODE CIRCUITS



Visualization of aliasing



Signals of different frequency and the same samples



Ideally one wants to eliminate frequency components higher than twice the sampling frequency and make sure that all useful frequencies are properly sampled

Design specification

half-power frequency at 100 Hz

Simplifying assumption

Infinite input resistance (no load on RC circuit)

$$\frac{V_o}{V_{in}} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

$$f_P = \frac{1}{2\pi RC} = 100 \text{ Hz}$$

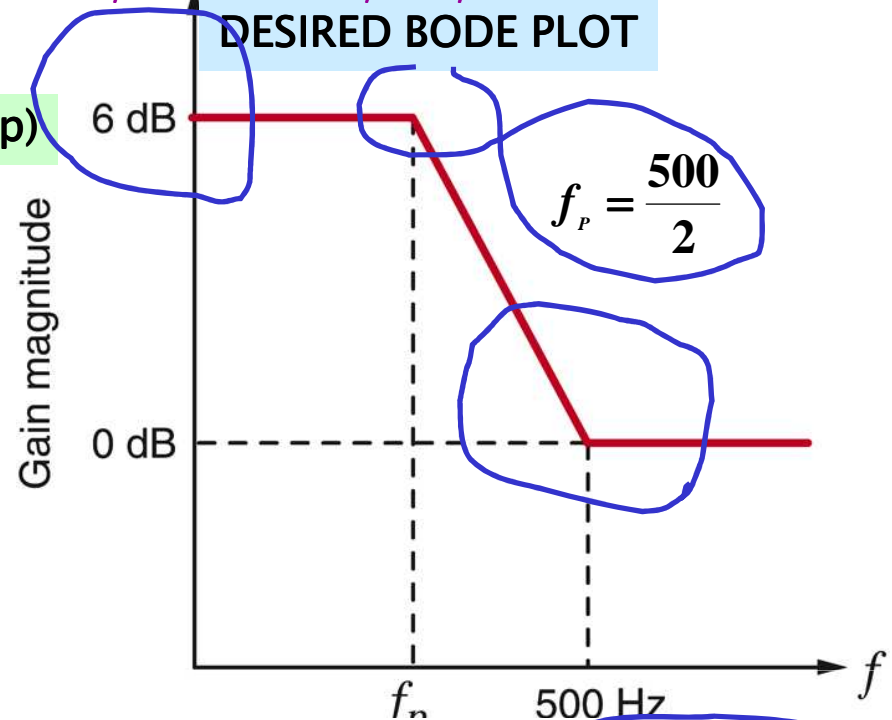
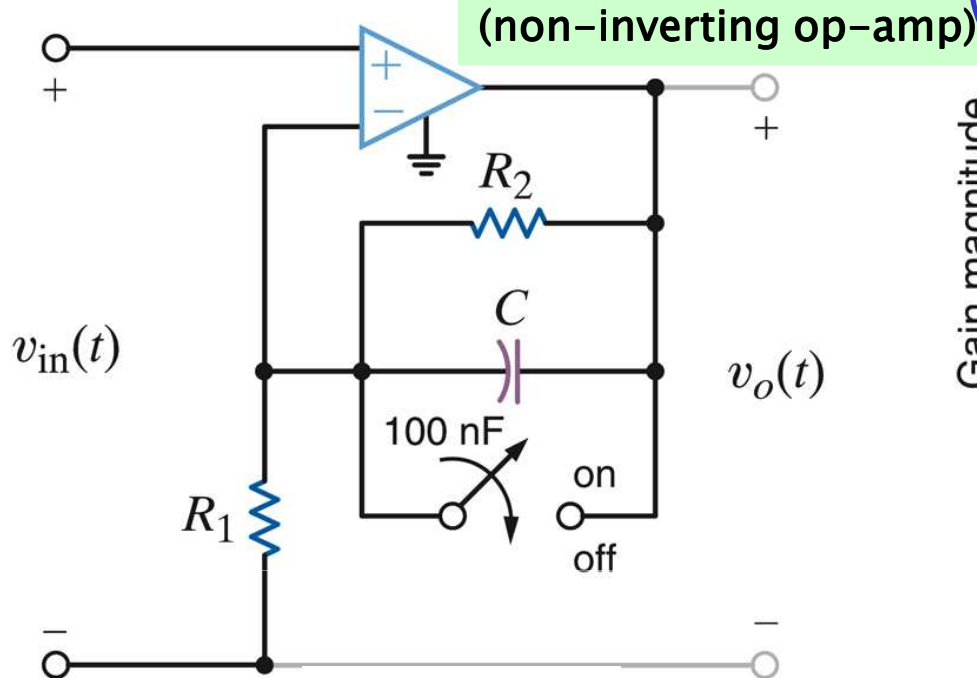
Design equation

somewhat arbitrarily choose C at 100 nF

$$\therefore R = 15.9k\Omega$$



DESIGN EXAMPLE "BASS-BOOST" AMPLIFIER



OPEN SWITCH

$$\frac{V_o}{V_{in}} = 1 + \frac{Z_2}{Z_1}$$

$$Z_1 = R_1 \quad Z_2 = R_2 // \frac{1}{j\omega C} = \frac{R_2}{1 + j\omega R_2 C}$$

$$\frac{V_o}{V_{in}} = \left[\frac{R_1 + R_2}{R_1} \right] \left[\frac{1 + j\omega R_P C}{1 + j\omega R_2 C} \right]$$

$$\frac{V_o}{V_{in}}(j0) = \frac{R_1 + R_2}{R_1} = 2 \quad (6\text{dB})$$

$\kappa_1 = \kappa_2$ and $\kappa_P = R_1/2$

$$\frac{1}{R_P C} = 2\pi(500) \Rightarrow R_P = \frac{1}{1000\pi C} = 3.18 \text{ k}\Omega$$

$$\frac{V_o}{V_{in}} = 1 + \frac{1 + j\omega R_2 C}{R_1} = \frac{R_1 + R_2 + j\omega R_1 R_2 C}{R_1(1 + j\omega R_2 C)}$$

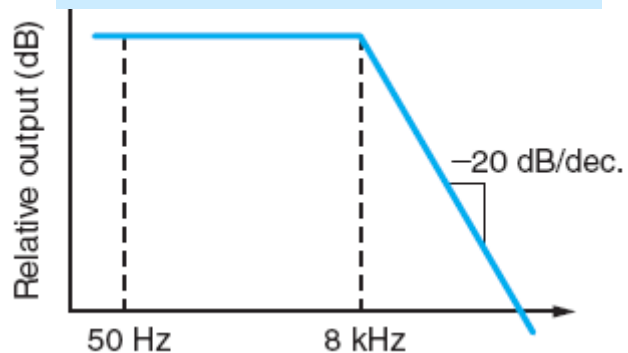
Switch closed??



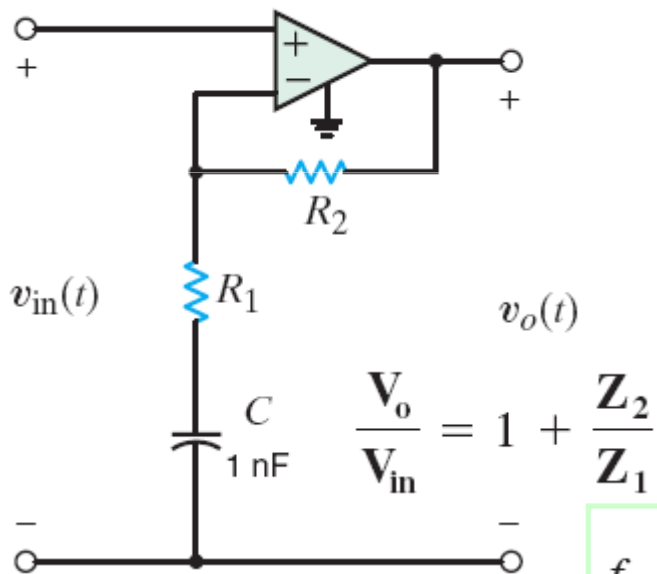
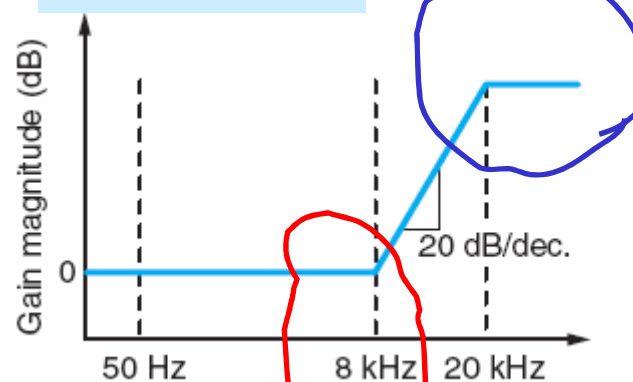
DESIGN EXAMPLE

TREBLE BOOST

Original player response



Desired boost



$$\frac{V_o}{V_{in}} = 1 + \frac{R_2}{R_1 + \frac{1}{j\omega C}} = \frac{1 + j\omega C(R_1 + R_2)}{1 + j\omega CR_1}$$

$$f_P = 2 \times 10^4 = \frac{1}{2\pi CR_1} \Rightarrow R_1 = 7.96 \text{ k}\Omega$$

Design equations

$$f_Z = 8 \times 10^3 = \frac{1}{2\pi C(R_1 + R_2)} \Rightarrow R_1 + R_2 = 19.9 \text{ k}\Omega$$

Proposed boost circuit

Non-inverting amplifier

Filters

